# CLASSIFICATION OF TORIC LOG DEL PEZZO SURFACES HAVING PICARD NUMBER 1 AND INDEX $\leq 3$

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*To Professor Friedrich Hirzebruch on the occasion of his eightieth birthday*

Abstract. Toric log Del Pezzo surfaces with Picard number 1 have been completely classified whenever their index is $\leq 2$. In this paper we extend the classification for those having index 3. We prove that, up to isomorphism, there are exactly 18 surfaces of this kind.

## 1. Introduction

Smooth compact toric surfaces belong to the basics in the framework of toric geometry. They are rational surfaces (i.e., of Kodaira dimension $-\infty$) defined by 2-dimensional complete fans which are composed of basic cones, and can therefore be studied by means of handy combinatorics (see [13, Theorem 1.28, pp. 42-43]). Of course, unlike the smooth compact complex surfaces having Kodaira dimension $\geq 0$, they do not possess uniquely determined *minimal models*. Nevertheless, the set of their minimal models consists of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ together with the *Hirzebruch surfaces*

$$\mathbb{F}_\kappa := \left\{ ([z_0 : z_1 : z_2], [t_1 : t_2]) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \;\middle|\; z_1 t_1^\kappa = z_2 t_2^\kappa \right\}, \quad \kappa \in \mathbb{Z}_{\geq 0},$$

for $\kappa \neq 1$ (cf. [10], [8, §2.5], [13, §1.7]), and it is known how one can pass from one minimal model to another by a finite succession of *elementary transformations*.

In contrast to this classical point of view, taking into account the fact that the *anti-Kodaira dimension* of smooth compact toric surfaces is 2, and switching to the so-called *antiminimal* and *anticanonical models* (in the sense of Sakai [15, §7], [16, Appendix]), one obtains surfaces which are *uniquely determined* up to isomorphism. However, since these models are mostly *singular*, in order to follow this choice we need a more systematic study of *singular* compact toric surfaces.

A graph-theoretic method of classifying (not necessarily smooth) compact toric surfaces *up to isomorphism* (generalizing Oda's graphs [13, pp. 44-46]) has been proposed in [5, §5]: Two compact toric surfaces are isomorphic to each other if and only if their vertex singly- and edge doubly-weighted circular graphs (WVE$^2$C-*graphs*, for short) are isomorphic (see below Theorem 4.4).

In addition, by [14, Theorem 4.3, pp. 398-399] the anticanonical models of smooth compact toric surfaces have to be log Del Pezzo surfaces. (A compact complex surface $X$ with at worst log terminal singularities, i.e., quotient singularities, is called *log Del Pezzo surface* if its anticanonical divisor $-K_X$ is a $\mathbb{Q}$-Cartier ample divisor. The *index* of such a surface is defined to be the smallest positive integer $\ell$

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for which $\ell K_X$ is a Cartier divisor. The family of log Del Pezzo surfaces of fixed index $\ell$ is known to be bounded (see [2, Theorem 2.1, p. 332])).

Consequently, it seems to be rather interesting to classify toric log Del Pezzo surfaces of given index $\ell$ up to isomorphism. A first attempt to understand the combinatorial complexity of this classification problem includes naturally the investigation of the case in which the Picard number $\rho(X_\Delta) := \mathrm{rank}(\mathrm{Pic}(X_\Delta))$ of surfaces $X_\Delta$ of this kind (associated to complete fans $\Delta$ in $\mathbb{R}^2$) equals 1. In this case, $X_\Delta$'s turn out to be weighted projective planes or quotients thereof by a finite abelian group. Let us first recall what is known for indices $\ell \leq 2$:

**Theorem 1.1.** *Up to isomorphism, there are exactly* 5 *toric log del Pezzo surfaces with Picard number* 1 *and index* $\ell = 1$, *namely*

| No. | (i) | (ii) | (iii) | (iv) | (v) |
|---|---|---|---|---|---|
| $X_\Delta$ | $\mathbb{P}^2_{\mathbb{C}}$ | $\mathbb{P}^2_{\mathbb{C}}/(\mathbb{Z}/3\mathbb{Z})$ | $\mathbb{P}^2_{\mathbb{C}}(1,1,2)$ | $\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/2\mathbb{Z})$ | $\mathbb{P}^2_{\mathbb{C}}(1,2,3)$ |

*whose* WVE$^2$C*-graphs are illustrated in* [5, Figure 8, p. 108].

**Theorem 1.2.** *Up to isomorphism, there are exactly* 7 *toric log del Pezzo surfaces with Picard number* 1 *and index* $\ell = 2$, *namely*

| No. | $X_\Delta$ | No. | $X_\Delta$ |
|---|---|---|---|
| (i) | $\mathbb{P}^2_{\mathbb{C}}(1,1,4)$ | (iv) | $\mathbb{P}^2_{\mathbb{C}}(1,2,3)/(\mathbb{Z}/2\mathbb{Z})$ |
| (ii) | $\mathbb{P}^2_{\mathbb{C}}(1,4,5)$ | (v) | $\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/4\mathbb{Z})$ |
| (iii) | $\mathbb{P}^2_{\mathbb{C}}(1,3,8)$ | (vi) | $\mathbb{P}^2_{\mathbb{C}}(1,2,1)/(\mathbb{Z}/4\mathbb{Z})$ |
| | | (vii) | $\mathbb{P}^2_{\mathbb{C}}(1,1,4)/(\mathbb{Z}/3\mathbb{Z})$ |

*whose* WVE$^2$C*-graphs are illustrated in* [5, Figure 11, p. 111].

In the present paper we extend these results also for index 3 by the following:

**Theorem 1.3.** *Up to isomorphism, there are exactly* 18 *toric log del Pezzo surfaces with Picard number* 1 *and index* $\ell = 3$, *namely*

| No. | $X_\Delta$ | No. | $X_\Delta$ |
|---|---|---|---|
| (i) | $\mathbb{P}^2_{\mathbb{C}}(1,1,3)$ | (x) | $\mathbb{P}^2_{\mathbb{C}}(1,5,9)$ |
| (ii) | $\mathbb{P}^2_{\mathbb{C}}(1,3,4)$ | (xi) | $\mathbb{P}^2_{\mathbb{C}}(1,2,9)$ |
| (iii) | $\mathbb{P}^2_{\mathbb{C}}(2,3,5)$ | (xii) | $\mathbb{P}^2_{\mathbb{C}}(1,2,3)/(\mathbb{Z}/3\mathbb{Z})$ |
| (iv) | $\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/3\mathbb{Z})$ | (xiii) | $\mathbb{P}^2_{\mathbb{C}}(1,1,2)/(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/3\mathbb{Z})$ |
| (v) | $\mathbb{P}^2_{\mathbb{C}}(1,1,6)$ | (xiv) | $\mathbb{P}^2_{\mathbb{C}}(1,1,6)/(\mathbb{Z}/2\mathbb{Z})$ |
| (vi) | $\mathbb{P}^2_{\mathbb{C}}(1,6,7)$ | (xv) | $\mathbb{P}^2_{\mathbb{C}}(1,4,15)$ |
| (vii) | $\mathbb{P}^2_{\mathbb{C}}(1,3,4)/(\mathbb{Z}/2\mathbb{Z})$ | (xvi) | $\mathbb{P}^2_{\mathbb{C}}(1,1,3)/(\mathbb{Z}/5\mathbb{Z})$ |
| (viii) | $\mathbb{P}^2_{\mathbb{C}}(1,2,3)/(\mathbb{Z}/3\mathbb{Z})$ | (xvii) | $\mathbb{P}^2_{\mathbb{C}}(1,2,9)/(\mathbb{Z}/2\mathbb{Z})$ |
| (ix) | $\mathbb{P}^2_{\mathbb{C}}/(\mathbb{Z}/9\mathbb{Z})$ | (xviii) | $\mathbb{P}^2_{\mathbb{C}}(1,1,6)/(\mathbb{Z}/4\mathbb{Z})$ |

*whose* WVE$^2$C*-graphs are illustrated below in* Figure 3.

The paper is organized as follows: In §2 we focus on the properties of the two non-negative, relatively prime integers $p = p_\sigma$ and $q = q_\sigma$ which parametrize the 2-dimensional, rational, strongly convex polyhedral cones $\sigma$, and recall how they are involved in Hirzebruch's minimal desingularization [11] of the 2-dimensional

cyclic quotient singularities $\mathrm{orb}(\sigma) \in \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2])$ for $q > 1$. In section 3 we give necessary and sufficient arithmetical conditions for the local indices $l = l_\sigma$ to be 1 or 3. Sections 4 and 5 are devoted to a detailed description of compact toric surfaces and of those which are log Del Pezzo surfaces. Some key-lemmas of combinatorial nature concerning compact toric surfaces with Picard number 1 are presented in §6. Based on the results of §3-§6 we explain how the classification method works in §7. The proof of Theorem 1.3 (which is somewhat longer than that of 1.1 and 1.2) follows in four steps (in §8-§11). The first three include the case by case determination of all "amissible" of triples of pairs $(p_i, q_i)$, $1 \leq i \leq 3$, so that the induced toric log Del Pezzo surfaces $X_\Delta$ with Picard number $\rho(X_\Delta) = 1$ have index $\ell = 3$. A minimal set of pairwise non-isomorphic surfaces of this kind is sorted out in the fourth step.

We use tools only from the classical toric geometry, adopting the standard terminology from [7], [8], and [13] (and mostly the notation introduced in [5]).

## 2. Two-dimensional toric singularities

Let $\sigma = \mathbb{R}_{\geq 0}\mathbf{n} + \mathbb{R}_{\geq 0}\mathbf{n}' \subset \mathbb{R}^2$ be a 2-dimensional, rational, strongly convex polyhedral cone. Without loss of generality we may assume that $\mathbf{n} = \binom{a}{b}$, $\mathbf{n}' = \binom{c}{d} \in \mathbb{Z}^2$, and that both $\mathbf{n}$ and $\mathbf{n}'$ are primitive elements of $\mathbb{Z}^2$, i.e., $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$.

**Lemma 2.1.** *Consider $\kappa, \lambda \in \mathbb{Z}$, such that $\kappa a - \lambda b = 1$. If $q := |ad - bc|$, and $p$ is the unique integer with*

$$0 \leq p < q \quad \text{and} \quad \kappa c - \lambda d \equiv p \,(\mathrm{mod}\ q)\,,$$

*then $\gcd(p, q) = 1$, and there exists a primitive element $\mathbf{n}'' = \binom{e}{g} \in \mathbb{Z}^2$, such that $\mathbf{n}' = p\mathbf{n} + q\mathbf{n}''$ and $\{\mathbf{n}, \mathbf{n}''\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.*

*Proof.* We define $\varepsilon := \mathrm{sign}(ad - bc)$ and write $\kappa c - \lambda d = \gamma q + p$, $\gamma \in \mathbb{Z}$. Setting $g := \varepsilon\kappa + \gamma b$ and $e := \varepsilon\lambda + \gamma a$, we get

$$gc - ed = \varepsilon\,(\kappa c - \lambda d) + \gamma\,(bc - ad) = \varepsilon\,(\gamma q + p) + \gamma\,(-\varepsilon q) = \varepsilon p,$$

i.e., $p = \varepsilon\,(gc - ed)$. On the other hand,

$$\det\left(\begin{smallmatrix} a & e \\ b & g \end{smallmatrix}\right) = ag - eb = \varepsilon\,(\kappa a - \lambda b) = \varepsilon,$$

which means that $\mathbf{n}''$ is primitive, $\{\mathbf{n}, \mathbf{n}''\}$ a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, and $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & e \\ b & g \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & p \\ 0 & q \end{smallmatrix}\right)$, i.e., $\mathbf{n}' = p\mathbf{n} + q\mathbf{n}''$, because

$$pa + qe = \varepsilon\,(gca - eda) + \varepsilon\,(ad - bc)\,e = c\varepsilon\,(ga - be) = c$$

and

$$pb + qg = \varepsilon\,(gcb - edb) + \varepsilon\,(ad - bc)\,g = \varepsilon d\,(ag - be) = d\ .$$

Since $\gcd(p, q)$ divides both $c$ and $d$, and $\gcd(c, d) = 1$, we obtain $\gcd(p, q) = 1$. $\square$

**Lemma 2.2.** *There is a linear map $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Phi(\mathbf{x}) := \Xi\,\mathbf{x}$, with $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, such that*

$$\Phi(\sigma) = \mathbb{R}_{\geq 0}\tbinom{1}{0} + \mathbb{R}_{\geq 0}\tbinom{p}{q}.$$

*Proof.* It it enough to define as $\Xi := \begin{pmatrix} \frac{\varepsilon(d-bp)}{q} & \frac{\varepsilon(ap-c)}{q} \\ -\varepsilon b & \varepsilon a \end{pmatrix}$. $\square$

Henceforth, we call $\sigma$ a $(p,q)$-*cone*. Denoting by $U_\sigma := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2])$ the affine toric variety associated to $\sigma$ (by means of the monoid $\sigma^\vee \cap \mathbb{Z}^2$, where $\sigma^\vee$ is the dual of $\sigma$) and by $\mathrm{orb}(\sigma)$ the single point being fixed under the usual action of the algebraic torus $\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*)$ on $U_\sigma$, it is easy to see that $U_\sigma \cong \mathbb{C}^2$ only if $q = 1$. (In this case, $\sigma$ is said to be a *basic cone*.) On the other hand, whenever $q > 1$ we have the following:

**Proposition 2.3.** $\mathrm{orb}(\sigma) \in U_\sigma$ *is a cyclic quotient singularity. In particular,*

$$U_\sigma \cong \mathbb{C}^2/G = \mathrm{Spec}(\mathbb{C}[z_1, z_2]^G),$$

*with* $G \subset \mathrm{GL}(2,\mathbb{C})$ *denoting the cyclic group* $G$ *of order* $q$ *which is generated by* $\mathrm{diag}(\zeta_q^{-p}, \zeta_q)$ $(\zeta_q := \exp(2\pi\sqrt{-1}/q))$ *and acts on* $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[z_1, z_2])$ *linearly and effectively.*

*Proof.* See [8, § 2.2, pp. 32-34] or [13, Proposition 1.24, p.30]. □

In fact, $U_\sigma$ is the toric variety $X_{\Delta_\sigma}$ defined by the fan

$$\Delta_\sigma := \{\sigma \text{ together with its faces}\},$$

and by Proposition 2.4 these two numbers $p = p_\sigma$ and $q = q_\sigma$ parametrize uniquely the isomorphism class of the germ $(U_\sigma, \mathrm{orb}(\sigma))$, up to replacement of $p$ by its socius $\widehat{p}$ (which corresponds just to the interchange of the coordinates). [The *socius* $\widehat{p}$ of $p$ is defined to be the uniquely determined integer, so that $0 \le \widehat{p} < q$, $\gcd(\widehat{p}, q) = 1$, and $p\,\widehat{p} \equiv 1 (\mathrm{mod}\ q)$.]

**Proposition 2.4.** *Let* $\sigma, \tau \subset \mathbb{R}^2$ *be two* 2*-dimensional, rational, stronly convex polyhedral cones. Then the following conditions are equivalent*:

(i) *There is a* $\mathbb{T}$*-equivariant isomorphism* $U_\sigma \cong U_\tau$ *mapping* $\mathrm{orb}(\sigma)$ *onto* $\mathrm{orb}(\tau)$.

(ii) *There exists a linear map* $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Phi(\mathbf{x}) := \Xi\,\mathbf{x}$, *with* $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, *such that* $\Phi(\sigma) = \tau$.

(iii) *For the numbers* $p_\sigma$, $p_\tau$, $q_\sigma$, $q_\tau$ *associated to* $\sigma, \tau$ (*by* Lemma 2.1) *we have* $q_\tau = q_\sigma$ *and either* $p_\tau = p_\sigma$ *or* $p_\tau = \widehat{p}_\sigma$.

*Proof.* For the equivalence (i)⇔(ii) see Ewald [7, Ch. VI, Thm. 2.11, pp. 222-223]. For proving (ii)⇔(iii) we may w.l.o.g. consider (by virtue of Lemma 2.2) the cones

$$\overline{\sigma} := \mathbb{R}_{\ge 0}\tbinom{1}{0} + \mathbb{R}_{\ge 0}\tbinom{p_\sigma}{q_\sigma} \quad \text{and} \quad \overline{\tau} := \mathbb{R}_{\ge 0}\tbinom{1}{0} + \mathbb{R}_{\ge 0}\tbinom{p_\tau}{q_\tau}$$

instead of $\sigma, \tau$.

(ii)⇒(iii): If there is a linear map $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\Phi(\mathbf{x}) := \Xi\,\mathbf{x}$, with $\Xi \in \mathrm{GL}_2(\mathbb{Z})$, such that $\Phi(\overline{\sigma}) = \overline{\tau}$, then either

$$\Phi\left(\tbinom{1}{0}\right) = \tbinom{1}{0} \quad \text{and} \quad \Phi\left(\tbinom{p_\sigma}{q_\sigma}\right) = \tbinom{p_\tau}{q_\tau}$$

or

$$\Phi\left(\tbinom{1}{0}\right) = \tbinom{p_\tau}{q_\tau} \quad \text{and} \quad \Phi\left(\tbinom{p_\sigma}{q_\sigma}\right) = \tbinom{1}{0}.$$

Thus, either

$$\Xi = \begin{pmatrix} 1 & \frac{p_\tau - p_\sigma}{q_\sigma} \\ 0 & \frac{q_\tau}{q_\sigma} \end{pmatrix} \qquad \text{or} \qquad \Xi = \begin{pmatrix} p_\tau & \frac{1 - p_\sigma\ p_\tau}{q_\sigma} \\ q_\tau & -\frac{p_\sigma\ q_\tau}{q_\sigma} \end{pmatrix}.$$

In the first case $\det(\Xi)$ has to be equal to 1, which means that $q_\sigma = q_\tau$ and $p_\tau - p_\sigma \equiv 0\ (\mathrm{mod}\ q_\sigma)$, i.e., $p_\tau = p_\sigma$ (because $0 \le p_\sigma, p_\tau \le q_\sigma = q_\tau$). In the second

case, $\det(\Xi) = -1$; hence, $q_\sigma = q_\tau$ and $1 - p_\sigma\, p_\tau \equiv 0 \pmod{q_\sigma}$, i.e., $p_\tau = \widehat{p}_\sigma$.

(iii)⇒(ii): If $q_\sigma = q_\tau$ and $p_\sigma = p_\tau$, we define $\Phi := \mathrm{id}_{\mathbb{R}^2}$. Otherwise, $q_\sigma = q_\tau$ and $p_\tau = \widehat{p}_\sigma$, and

$$\Phi\left(\mathbf{x}\right) := \left(\begin{array}{cc} p_\tau & \frac{1}{q_\sigma} - p_\sigma \\ q_\sigma & -p_\sigma \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right), \quad \forall \mathbf{x} = \binom{x_1}{x_2} \in \mathbb{R}^2,$$

is an $\mathbb{R}$-vector space isomorphism with the desired property. □

To construct the minimal desingularization of $U_\sigma$ for a $(p,q)$-cone

$$\sigma = \mathbb{R}_{\geq 0}\mathbf{n} + \mathbb{R}_{\geq 0}\mathbf{n}' \subset \mathbb{R}^2 \quad (\text{with } q > 1)$$

we consider the negative-regular continued fraction expansion of

$$\frac{q}{q-p} = [\![b_1, b_2, \ldots, b_s]\!] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \, b_{s-1} - \cfrac{1}{b_s}}} \,,$$

and define $\mathbf{u}_0 := \mathbf{n}$, $\mathbf{u}_1 := \frac{1}{q}((q-p)\mathbf{n} + \mathbf{n}')$, and lattice points $\{\mathbf{u}_j \,|\, 2 \leq j \leq s+1\}$ by the formulae

$$\mathbf{u}_{j+1} := b_j \mathbf{u}_j - \mathbf{u}_{j-1}, \quad \forall j \in \{1, \ldots, s\}.$$

It is easy to see that $\mathbf{u}_{s+1} = \mathbf{n}'$, and that the integers $b_j$ are $\geq 2$, for all indices $j \in \{1, \ldots, s\}$. Next, we subdivide $\sigma$ into $s+1$ smaller basic cones by introducing new rays passing through the points $\mathbf{u}_1, \ldots, \mathbf{u}_s$.

**Theorem 2.5** (Toric version of Hirzebruch's desingularization)**.** *The refinement*

$$\widetilde{\Delta}_\sigma := \{\{\mathbb{R}_{\geq 0}\, \mathbf{u}_j + \mathbb{R}_{\geq 0}\, \mathbf{u}_{j+1} \;\mid 0 \leq j \leq s\} \quad \text{together with their faces}\}$$

*of* $\Delta_\sigma := \{\sigma \text{ together with its faces}\}$ *consists of basic cones, is the coarsest refinement of* $\Delta_\sigma$ *with this property, and induces the* minimal $\mathbb{T}$-equivariant resolution $X_{\widetilde{\Delta}_\sigma} \longrightarrow X_{\Delta_\sigma} = U_\sigma$ *of the singular point* $\mathrm{orb}(\sigma)$. *Moreover, the exceptional divisor is* $E := \sum_{j=1}^{s} E_j$, *having*

$$E_j := \overline{\mathrm{orb}_{\widetilde{\Delta}_\sigma}(\mathbb{R}_{\geq 0}\, \mathbf{u}_j)} \; (\cong \mathbb{P}^1_{\mathbb{C}}), \quad \forall j \in \{1, \ldots, s\},$$

(*i.e., the closures of the* $\mathbb{T}$*-orbits of the new rays w.r.t.* $\widetilde{\Delta}_\sigma$) *as its components, with self-intersection number* $(E_j)^2 = -b_j$.

*Proof.* See Hirzebruch [11, pp. 15-20] who constructs $X_{\widetilde{\Delta}_\sigma}$ by resolving the unique singularity lying over $\mathbf{0} \in \mathbb{C}^3$ in the normalization of the hypersurface

$$\left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \middle|\; z_1^q - z_2 z_3^{q-p} = 0 \right\},$$

and Oda [13, pp. 24-30] for a proof which uses only the tools of toric geometry. □

## 3. Local indices

Let $\sigma \subset \mathbb{R}^2$ be a $(p,q)$-cone. We define the *local index* $l = l_\sigma$ of $\sigma$ to be the positive integer

$$l := \begin{cases} 1, & \text{if } q = 1, \\ \min\{k \in \mathbb{N} \mid k\,K(E) \text{ is a Cartier divisor}\}, & \text{if } q > 1, \end{cases} \tag{3.1}$$

where $K(E)$ denotes the *local canonical divisor* of $X_{\widetilde{\Delta}_\sigma}$ at $\text{orb}(\sigma)$ (in the sense of [5, p. 75]) w.r.t. the minimal resolution $X_{\widetilde{\Delta}_\sigma} \longrightarrow X_{\Delta_\sigma}$ of $\text{orb}(\sigma)$ constructed in Theorem 2.5. It can be shown that

$$l = \frac{q}{\gcd(q, p-1)}, \tag{3.2}$$

cf. [5, Note 3.19, p. 89, and Prop. 4.4, pp. 94-95], and that the self-intersection number of $K(E)$ equals

$$K(E)^2 = -\left(\frac{2-(p+\widehat{p})}{q} + \sum_{j=1}^{s}(b_j - 2)\right), \tag{3.3}$$

cf. [5, Corollary 4.6, p. 96]. For the proof of Theorem 1.3 we need to know under which restrictions on $p$ and $q$ we have $l \in \{1,3\}$.

**Lemma 3.1.** *If $\sigma \subset \mathbb{R}^2$ is a $(p,q)$-cone, then*

$$l = 1 \Longleftrightarrow \begin{cases} \text{either } p = 0 \text{ and } q = 1, \\ \text{or } p = 1 \text{ and } q \geq 2, \end{cases} \tag{3.4}$$

*Proof.* By (3.2), $l = 1 \Longleftrightarrow q = \gcd(q, q-p+1)$, and therefore $q \mid p-1$. Since $p - 1 < p < q$, $p$ and $q$ satisfy conditions (3.4). □

**Lemma 3.2.** *If $\sigma \subset \mathbb{R}^2$ is a $(p,q)$-cone, then*

$$l = 3 \Longleftrightarrow \begin{cases} \text{either } (p,q) \in A, \\ \text{or } (p,q) \in B, \end{cases} \tag{3.5}$$

*where*

$$A := \{(p,q) \in \mathbb{N} \times \mathbb{N} \mid q = 3(p-1), \quad p \geq 2, \quad 3 \nmid p\},$$

*and*

$$B := \left\{(p,q) \in \mathbb{N} \times \mathbb{N} \mid q = \frac{3}{2}(p-1), \quad p \text{ odd } \geq 5, \quad 3 \nmid p\right\}.$$

*Moreover, if $(p,q) \in A$ and $(p',q) \in B$, then*

$$p' = \widehat{p} \;(= \text{the socius of } p) \Longleftrightarrow pp' \equiv 1 (\text{mod } q) \Longleftrightarrow q \equiv 0 (\text{mod } 9).$$

*Proof.* $l = 3$ means that $q = 3m$, where $m := \gcd(q, p-1)$. Write $p - 1 = am$. Since $1 \leq p < q$, we have $a \in \{1,2\}$. Since $\gcd(p,q) = 1$, in the case in which $a = 1$, we get $\gcd(3m, m+1) = 1 \Longleftrightarrow \gcd(3,p) = 1 \Longleftrightarrow 3 \nmid p$, i.e. $(p,q) \in A$, whereas in the case in which $a = 2$, we get $\gcd(3m, 2m+1) = 1 \Longleftrightarrow \gcd(3,p) = 1 \Longleftrightarrow 3 \nmid p$, and $p$ odd $\geq 5$, i.e. $(p,q) \in B$. Hence, (3.5) is true. The last assertion can be verified easily. □

**Note 3.3.** It is worthwhile to take a closer look at the sets $A$ and $B$, and to the corresponding negative-regular continued fraction expansions.

Set $A$ :

| $p$ | 2 | *4* | 5 | *7* | 8 | *10* | 11 | *13* | 14 | *16* | 17 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | 3 | *9* | 12 | *18* | 21 | *27* | 30 | *36* | 39 | *45* | 48 | $\cdots$ |

• First case: Whenever $\boxed{9 \nmid q}$ we have $\widehat{p} = p$ and

$$\frac{q}{q-p} = \begin{cases} 3, & \text{if } p = 2,\ q = 3, \\ [\![2,4,2]\!]\,, & \text{if } p = 5,\ q = 12, \\ [\![2,3,\underbrace{2,...,2}_{\left(\frac{q-3}{9}-2\right)\text{-times}},3,2]\!], & \text{if } p \geq 8,\ q \geq 21. \end{cases}$$

• Second case: Whenever $\boxed{9 \mid q}$ we have $\widehat{p} = 2p - 1$ and

$$\frac{q}{q-p} = \begin{cases} [\![2,5]\!]\,, & \text{if } p = 4,\ q = 9, \\ [\![2,3,\underbrace{2,...,2}_{\left(\frac{q}{9}-2\right)\text{-times}},4]\!], & \text{if } p \geq 7,\ q \geq 18. \end{cases}$$

Set $B$ :

| $p$ | 5 | *7* | 11 | *13* | 17 | *19* | 23 | *25* | 29 | *31* | 35 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | 6 | *9* | 15 | *18* | 24 | *27* | 33 | *36* | 42 | *45* | 51 | $\cdots$ |

• First case: Whenever $\boxed{9 \nmid q}$ we have $\widehat{p} = p$ and

$$\frac{q}{q-p} = \begin{cases} 6, & \text{if } p = 5,\ q = 6, \\ [\![4,\underbrace{2,...,2}_{\left(\frac{q-6}{9}-1\right)\text{-times}},4]\!], & \text{if } p \geq 11,\ q \geq 15. \end{cases}$$

• Second case: Whenever $\boxed{9 \mid q}$ we have $\widehat{p} = \frac{1}{2}(p+1)$ and

$$\frac{q}{q-p} = \begin{cases} [\![5,2]\!]\,, & \text{if } p = 7,\ q = 9, \\ [\![4,\underbrace{2,...,2}_{\left(\frac{q}{9}-2\right)\text{-times}},3,2]\!], & \text{if } p \geq 13,\ q \geq 18. \end{cases}$$

These continued fraction expansions will be useful in what follows in §7.

## 4. Compact toric surfaces

Every compact toric surface is a 2-dimensional toric variety $X_\Delta$ associated to a *complete* fan $\Delta$ in $\mathbb{R}^2$, i.e., a fan having 2-dimensional cones as maximal cones and whose support $|\Delta|$ is the entire $\mathbb{R}^2$ (see [13, Theorem 1.11, p. 16]). Consider a complete fan $\Delta$ in $\mathbb{R}^2$ and suppose that

$$\sigma_i = \mathbb{R}_{\geq 0}\mathbf{n}_i + \mathbb{R}_{\geq 0}\mathbf{n}_{i+1}, \quad i \in \{1, \ldots, \nu\}, \tag{4.1}$$

are its 2-dimensional cones (with $\nu \geq 3$ and $\mathbf{n}_i$ primitive for all $i \in \{1, \ldots, \nu\}$), enumerated in such a way that $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ go *anticlockwise* around the origin exactly once in this order (under the usual convention: $\mathbf{n}_{\nu+1} := \mathbf{n}_1$, $\mathbf{n}_0 := \mathbf{n}_\nu$). Since $\Delta$ is simplicial, the Picard number $\rho(X_\Delta)$ of $X_\Delta$ (i.e., the rank of its Picard group $\mathrm{Pic}(X_\Delta)$) equals

$$\rho(X_\Delta) = \ \nu - 2, \tag{4.2}$$

(see [8, p. 65]). Now suppose that $\sigma_i$ is a $(p_i, q_i)$-cone for all $i \in \{1, \ldots, \nu\}$ and introduce the notation

$$I_\Delta := \{ i \in \{1, \ldots, \nu\} \mid q_i > 1 \}, \quad J_\Delta := \{ i \in \{1, \ldots, \nu\} \mid q_i = 1 \}, \tag{4.3}$$

to separate the indices corresponding to non-basic from those corresponding to basic cones. By [13, Theorem 1.10, p. 15] the singular locus of $X_\Delta$ equals

$$\mathrm{Sing}(X_\Delta) = \{ \mathrm{orb}(\sigma_i) \mid i \in I_\Delta \},$$

and its subset

$$\{\mathrm{orb}(\sigma_i) \mid i \in \breve{I}_\Delta\}, \quad \text{with} \quad \breve{I}_\Delta := \{ i \in I_\Delta \mid p_i = 1 \}, \tag{4.4}$$

constitutes the set of the *Gorenstein singularities* of $X_\Delta$. For all $i \in I_\Delta$ write

$$\frac{q_i}{q_i - p_i} = \left[\!\left[ b_1^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)} \right]\!\right] \tag{4.5}$$

and, in accordance with what is already mentioned for a single 2-dimensional non-basic cone in §2, define

$$\mathbf{u}_1^{(i)} := \mathbf{n}_i, \quad \mathbf{u}_1^{(i)} := \frac{1}{q_i}((q_i - p_i)\mathbf{n}_i + \mathbf{n}_{i+1}),$$

and

$$\mathbf{u}_{j+1}^{(i)} = b_j^{(i)} \mathbf{u}_j^{(i)} - \mathbf{u}_{j-1}^{(i)}, \quad \forall j \in \{1, \ldots, s_i\} \ \ (\text{with } \mathbf{u}_{s_i+1}^{(i)} = \mathbf{n}_{i+1}).$$

By construction, the proper birational map $f : X_{\widetilde{\Delta}} \longrightarrow X_\Delta$ induced by the refinement

$$\widetilde{\Delta} := \left\{ \begin{array}{c} \text{the cones } \{ \sigma_i \mid i \in J_\Delta \} \text{ and} \\ \left\{ \mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(i)} + \mathbb{R}_{\geq 0}\, \mathbf{u}_{j+1}^{(i)} \ \middle| \ i \in I_\Delta, \ j \in \{0, 1, \ldots, s_i\} \right\}, \\ \text{together with their faces} \end{array} \right\}.$$

of the fan $\Delta$ is the *minimal desingularization* of $X_\Delta$. Defining

$$\begin{cases} E_j^{(i)} := \ \overline{\mathrm{orb}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\, \mathbf{u}_j^{(i)})}, & \forall i \in I_\Delta \ \text{ and } \ \forall j \in \{1, 2, \ldots, s_i\}, \\ \overline{C}_i := \overline{\mathrm{orb}_{\widetilde{\Delta}}(\mathbb{R}_{\geq 0}\, \mathbf{n}_i)}, & \forall i \in \{1, 2, \ldots, \nu\}, \end{cases}$$

one observes that $\overline{C}_i$ is the *strict transform* of $C_i := \overline{\mathrm{orb}_\Delta(\mathbb{R}_{\geq 0}\, \mathbf{n}_i)}$ w.r.t. $f$,

$$E^{(i)} := \sum_{j=1}^{s_i} E_j^{(i)}$$

the *exceptional divisor* replacing $\mathrm{orb}(\sigma_i)$ via $f$ (with $(E_j^{(i)})^2 = -b_j^{(i)}$, $\forall i \in I_\Delta$ and $\forall j \in \{1, 2, \ldots, s_i\}$), and

$$K_{X_{\widetilde{\Delta}}} - f^* K_{X_\Delta} = \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} K(E^{(i)}) \tag{4.6}$$

the *discrepancy divisor* w.r.t. $f$. (By $K_{X_\Delta}, K_{X_{\widetilde{\Delta}}}$ we denote the canonical divisors of $X_\Delta$ and $X_{\widetilde{\Delta}}$, respectively.)

**Proposition 4.1.** *The Picard number of $X_{\widetilde{\Delta}}$ equals*

$$\rho(X_{\widetilde{\Delta}}) = \sum_{i \in I_\Delta} s_i + (\nu - 2) = 10 - K^2_{X_\Delta} - \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} K(E^{(i)})^2. \tag{4.7}$$

*Proof.* The first equality follows from (4.2) and from the fact that

$$\rho(X_{\widetilde{\Delta}}) = \rho(X_\Delta) + \sharp\{\text{exceptional prime divisors w.r.t. } f\}.$$

(4.6) implies

$$K^2_{X_{\widetilde{\Delta}}} = K^2_{X_\Delta} + \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} K(E^{(i)})^2.$$

Substituting this expression for $K^2_{X_{\widetilde{\Delta}}}$ into Noether's formula

$$K^2_{X_{\widetilde{\Delta}}} = 10 - \rho(X_{\widetilde{\Delta}}),$$

we obtain the second equality of (4.7). □

**Definition 4.2** (The additional characteristic numbers $r_i$)**.** For every $i \in \{1, .., \nu\}$ we introduce integers $r_i$ *uniquely determined* by the conditions:

$$r_i \mathbf{n}_i = \begin{cases} \mathbf{u}^{(i-1)}_{s_{i-1}} + \mathbf{u}^{(i)}_1, & \text{if } i \in I'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{u}^{(i)}_1, & \text{if } i \in I''_\Delta, \\ \mathbf{u}^{(i-1)}_{s_{i-1}} + \mathbf{n}_{i+1}, & \text{if } i \in J'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{n}_{i+1}, & \text{if } i \in J''_\Delta, \end{cases} \tag{4.8}$$

where

$$I'_\Delta := \{ i \in I_\Delta \mid q_{i-1} > 1 \}, \quad I''_\Delta := \{ i \in I_\Delta \mid q_{i-1} = 1 \},$$

and

$$J'_\Delta := \{ i \in J_\Delta \mid q_{i-1} > 1 \}, \quad J''_\Delta := \{ i \in J_\Delta \mid q_{i-1} = 1 \},$$

with $I_\Delta, J_\Delta$ as in (4.3).

By [5, Lemma 4.3], for $i \in \{1, \dots, \nu\}$, $-r_i$ is nothing but the self-intersection number $\overline{C}_i^2$ of the strict transform $\overline{C}_i$ of $C_i$ w.r.t. $f$. The triples $(p_i, q_i, r_i)$, $i \in \{1, 2, \dots, \nu\}$, are used to define the WVE$^2$C-graph $\mathfrak{G}_\Delta$.

**Definition 4.3.** A *circular graph* is a plane graph whose vertices are points on a circle and whose edges are the corresponding arcs (on this circle, each of which connects two consecutive vertices). We say that a circular graph $\mathfrak{G}$ is $\mathbb{Z}$*-weighted at its vertices* and *double* $\mathbb{Z}$*-weighted at its edges* (and call it WVE$^2$C-*graph*, for short) if it is accompanied by two maps

$$\{\text{Vertices of } \mathfrak{G}\} \longmapsto \mathbb{Z}, \ \{\text{Edges of } \mathfrak{G}\} \longmapsto \mathbb{Z}^2,$$

assigning to each vertex an integer and to each edge a pair of integers, respectively. To every complete fan $\Delta$ in $\mathbb{R}^2$ (as described above) we associate an anticlockwise directed WVE$^2$C-graph $\mathfrak{G}_\Delta$ with

$$\{\text{Vertices of } \mathfrak{G}_\Delta\} = \{\mathbf{v}_1, \dots, \mathbf{v}_\nu\} \ \text{ and } \ \{\text{Edges of } \mathfrak{G}_\Delta\} = \{\overline{\mathbf{v}_1\mathbf{v}_2}, \dots, \overline{\mathbf{v}_\nu\mathbf{v}_1}\},$$

($\mathbf{v}_{\nu+1} := \mathbf{v}_1$), by defining its "weights" as follows:

$$\mathbf{v}_i \longmapsto -r_i, \quad \overline{\mathbf{v}_i\mathbf{v}_{i+1}} \longmapsto (p_i, q_i), \ \forall i \in \{1, \dots, \nu\}.$$

The *reverse graph* $\mathfrak{G}_\Delta^{\mathrm{rev}}$ of $\mathfrak{G}_\Delta$ is the directed WVE$^2$C-graph which is obtained by changing the double weight $(p_i, q_i)$ of the edge $\overline{\mathbf{v}_i \mathbf{v}_{i+1}}$ into $(\widehat{p}_i, q_i)$ and reversing the initial anticlockwise direction of $\mathfrak{G}_\Delta$ into clockwise direction (see Figure 1).

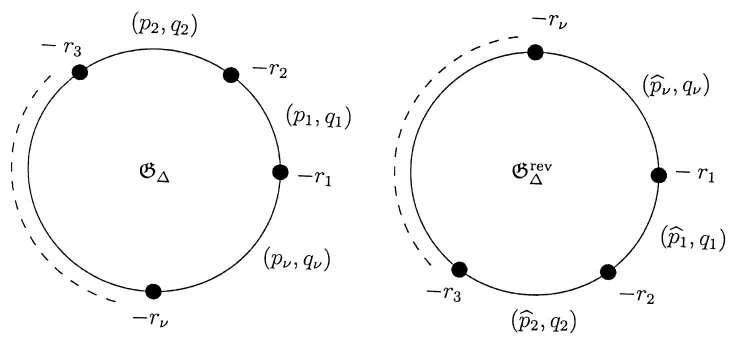



FIGURE 1.

**Theorem 4.4** (Classification up to isomorphism)**.** *Let $\Delta$, $\Delta'$ be two complete fans in $\mathbb{R}^2$. Then the following conditions are equivalent*:

(i) *The compact toric surfaces $X_\Delta$ and $X_{\Delta'}$ are isomorphic.*

(ii) *Either $\mathfrak{G}_{\Delta'} \overset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta$ or $\mathfrak{G}_{\Delta'} \overset{\mathrm{gr.}}{\cong} \mathfrak{G}_\Delta^{\mathrm{rev}}$.*

Here "$\overset{\mathrm{gr.}}{\cong}$" indicates graph-theoretic isomorphism (i.e., a bijection between the sets of vertices which preserves the corresponding weights). For further details and for the proof of Theorem 4.4 the reader is referred to [5, §5].

## 5. Toric log Del Pezzo surfaces

Let $X_\Delta$ be a compact toric surface defined by a complete fan $\Delta$ in $\mathbb{R}^2$ having (4.1) as its 2-dimensional cones. (Throughout this section we maintain the notation introduced in §4.) It is known that $X_\Delta$ is a log Del Pezzo surface if and only if the minimal generators $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ of the rays of $\Delta$ are vertices of a lattice polygon $Q_\Delta$ (cf. [5, Remark 6.7, p. 107]).

**Definition 5.1.** A polygon $Q \subset \mathbb{R}^2$ is called *LDP-polygon* if it contains the origin in its interior, and its vertices are primitive elements of $\mathbb{Z}^2$.

In fact, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of toric log Del Pezzo} \\ \text{surfaces} \end{array} \right\} \ni [X_\Delta] \longmapsto [Q_\Delta] \in \left\{ \begin{array}{c} \text{lattice-equivalence} \\ \text{classes} \\ \text{of LDP-polygons} \end{array} \right\}.$$

Indeed, if $X_\Delta \cong X_{\Delta'}$, then by Theorem 4.4 there exists a unimodular trasformation $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $\Phi(Q_\Delta) = Q_{\Delta'}$. The inverse of the above correspondence is given by mapping the lattice-equivalence class $[Q]$ of any LDP-polygon $Q$ onto $\left[X_{\Delta_Q}\right]$, where

$$\Delta_Q := \{\text{the cones } \mathbb{R}_{\geq 0} F \ \text{ together with their faces} \mid F \in \mathcal{F}(Q)\},$$

and $\mathcal{F}(Q) := \{\text{facets (edges) of } Q\}$. (If $Q$ is an LDP-polygon,

$$\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \Phi(\mathbf{x}) := \Xi\,\mathbf{x}, \ \forall \mathbf{x} \in \mathbb{R}^2, \ \text{with } \Xi \in \mathrm{GL}_2(\mathbb{Z}),$$

and $Q' := \Phi(Q)$, then $\mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr.}}{\cong} \mathfrak{G}_{\Delta_Q}$ whenever $\det(\Xi) = 1$, and $\mathfrak{G}_{\Delta_{Q'}} \overset{\text{gr.}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_Q}$ whenever $\det(\Xi) = -1$.)

Therefore, the classification of toric log Del Pezzo surfaces (up to isomorphism) is equivalent to the classification of LDP-polygons (up to unimodular transformations). Since the number of lattice-equivalence classes of LDP-polygons $Q_\Delta$ for all those $X_\Delta$'s having *fixed* index $\ell$ (with $\ell$ as defined in §1) is *finite*, as it follows from results appearing in [1], [3], [9] and [12], it is reasonable (for any systematic approach to the classification problem) to focus on $\ell$. By (3.1), (3.2), (4.4) and (4.6) we obtain:

**Lemma 5.2.** *The index $\ell$ of a toric log Del Pezzo surface $X_\Delta$ equals*

$$\ell = \begin{cases} \mathrm{lcm}\{ l_i \mid i \in I_\Delta \} \ (= \mathrm{lcm}\{ l_i \mid i \in I_\Delta \smallsetminus \breve{I}_\Delta \}), & \text{if } I_\Delta \neq \varnothing, \\ 1, & \text{if } I_\Delta = \varnothing, \end{cases} \tag{5.1}$$

*where $l_i = l_{\sigma_i}$ is the local index of $\sigma_i$ (cf. (3.2)).*

**Remark 5.3.** In geometric terms, $\ell = \min\{ k \in \mathbb{N} | kQ^*_\Delta \text{ is a lattice polygon}\}$, where $Q^*_\Delta$ denotes the *polar* of the polygon $Q_\Delta$. In other words, $\ell$ equals the least common multiple of the (smallest) denominators of the (rational) coordinates of the vertices of $Q^*_\Delta$. Moreover, for $\ell \geq 2$, $\nu = \sharp\{\text{vertices of } Q_\Delta\} \leq 4\ell + 1$ (see [6, Lemma 3.1]).

**Proposition 5.4.** *For any toric log Del Pezzo surface $X_\Delta$ of index $\ell \geq 1$ the following inequality holds*:

$$\sum_{i \in I_\Delta} s_i \leq 12 - \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} K(E^{(i)})^2 - \left(1 + \frac{1}{\ell}\right)\nu. \tag{5.2}$$

*Proof.* (5.2) follows from (4.7) and $K^2_{X_\Delta} \geq \frac{\nu}{\ell}$ (see [6, proof of Lemma 3.2]). □

An additional necessary condition for a compact toric surface $X_\Delta$ to be log Del Pezzo is dictated by the *convexity* of the necessarily existing LDP-polygon $Q_\Delta$:

**Proposition 5.5.** *For any toric log Del Pezzo surface $X_\Delta$ of index $\ell \geq 2$ we have*

$$\sum_{i \in \breve{I}_\Delta} q_i \leq \left( \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} \left(1 - \frac{2}{l_i}\right) q_i \right) - (\nu - \sharp(I_\Delta)) + 8. \tag{5.3}$$

*Proof.* Since

$$\sharp(\mathrm{int}(\mathrm{conv}(\{\mathbf{n}_i, \mathbf{n}_{i+1}\}) \cap \mathbb{Z}^2) = \ \gcd(q_i, p_i - 1) - 1, \quad \forall i \in \{1, \ldots, \nu\},$$

we obtain

$$\sharp(\partial Q_\Delta \cap \mathbb{Z}^2) = \nu + \sum_{i=1}^{\nu} \sharp(\mathrm{int}(\mathrm{conv}(\{\mathbf{n}_i, \mathbf{n}_{i+1}\}) \cap \mathbb{Z}^2) = \sum_{i=1}^{\nu} \gcd(q_i, p_i - 1). \tag{5.4}$$

($\partial$, int, and conv are used as abbreviations for boundary, interior, and convex hull, respectively.) Furthermore, since

$$\mathrm{area}(Q_\Delta) = \sum_{i=1}^{\nu} \mathrm{area}(\mathrm{conv}(\{\mathbf{0}, \mathbf{n}_i, \mathbf{n}_{i+1}\})) = \frac{1}{2}\left(\sum_{i=1}^{\nu} q_i\right),$$

using Pick's formula (cf. [8, p. 113], [13, p. 101]):

$$\sharp(Q_\Delta \cap \mathbb{Z}^2) = \mathrm{area}(Q_\Delta) + \frac{1}{2}\sharp(\partial Q_\Delta \cap \mathbb{Z}^2) + 1,$$

we get

$$\sharp(\mathrm{int}(Q_\Delta) \cap \mathbb{Z}^2) = \frac{1}{2}\left(\sum_{i=1}^{\nu}(q_i - \gcd(q_i, p_i - 1))\right) + 1. \tag{5.5}$$

Finally, since $\ell \geq 2$, Scott's inequality [17] can be written as

$$\sharp(\partial Q_\Delta \cap \mathbb{Z}^2) < 2\,\sharp(\mathrm{int}(Q_\Delta) \cap \mathbb{Z}^2) + 7. \tag{5.6}$$

By (5.4), (5.5), (5.6) and (3.2) we infer that

$$\sum_{i=1}^{\nu}\left(\frac{2}{l_i} - 1\right) q_i \leq 8,$$

which can be rewritten (by keeping the involved $q_i$'s with non-negative coefficients) in the form (5.3). □

## 6. Compact toric surfaces with Picard number 1

By virtue of (4.2) the compact toric surfaces with Picard number 1 are defined by complete fans $\Delta$ in $\mathbb{R}^2$ with exactly three 2-dimensional cones. Let $\Delta$ be a complete fan of this kind and

$$\sigma_1 = \mathbb{R}_{\geq 0}\mathbf{n}_1 + \mathbb{R}_{\geq 0}\mathbf{n}_2, \quad \sigma_2 = \mathbb{R}_{\geq 0}\mathbf{n}_2 + \mathbb{R}_{\geq 0}\mathbf{n}_3, \quad \sigma_3 = \mathbb{R}_{\geq 0}\mathbf{n}_3 + \mathbb{R}_{\geq 0}\mathbf{n}_1, \tag{6.1}$$

be its 2-dimensional cones, with $\mathbf{n}_i$ primitive and $\sigma_i$ a $(p_i, q_i)$-cone for $i \in \{1, 2, 3\}$.

**Lemma 6.1.** *$X_\Delta$ is isomorphic to the quotient space $\mathbb{P}^2_{\mathbb{C}}(q_1, q_2, q_3)/H_\Delta$, where $H_\Delta$ is a finite abelian group of order* $\gcd(q_1, q_2, q_3)$.

*Proof.* Since $q_i = |\det(\mathbf{n}_i, \mathbf{n}_{i+1})|$ for $i \in \{1, 2, 3\}$, using Cramer's rule we obtain

$$q_1\mathbf{n}_3 + q_2\mathbf{n}_1 + q_3\mathbf{n}_2 = \mathbf{0}.$$

By [4, Proposition 4.7, p. 224] we have $X_\Delta \cong \mathbb{P}^2_{\mathbb{C}}(q_1, q_2, q_3)/H_\Delta$, where $H_\Delta$ is a group isomorphic to $\mathbb{Z}^2/(\oplus_{i=1}^3 \mathbb{Z}\mathbf{n}_i)$. By

$$|H_\Delta| = \sharp\left(\{\text{fundamental perallelepiped of } \oplus_{i=1}^3 \mathbb{Z}\mathbf{n}_i\} \cap \mathbb{Z}^2\right) = \det(\oplus_{i=1}^3 \mathbb{Z}\mathbf{n}_i),$$

and the fact that $\det(\oplus_{i=1}^3 \mathbb{Z}\mathbf{n}_i) = \gcd(q_1, q_2, q_3)$, the assertion is true. □

Since we are interested in describing $X_\Delta$ *up to isomorphism* (cf. Lemma 2.2 and Theorem 4.4) we may henceforth assume, without loss of generality, that $\mathbf{n}_1 = \binom{1}{0}$ and $\mathbf{n}_2 = \binom{p_1}{q_1}$. As all cones of $\Delta$ are strongly convex, $\mathbf{n}_3$ belongs (as shown in Figure 2) necessarily to the set

$$\mathcal{M} := \left\{ \binom{x}{y} \in \mathbb{Z}^2 \;\middle|\; \tfrac{q_1}{p_1}x < y < 0 \right\}.$$

**Lemma 6.2.** *We have*

$$\mathbf{n}_3 = \binom{-(q_2 + p_1 q_3)/q_1}{-q_3}, \tag{6.2}$$

*and therefore $q_1 \mid q_2 + p_1 q_3$ and $\gcd((q_2 + p_1 q_3)/q_1, q_3) = 1$. Moreover,*

$$q_1 q_2 \mid \widehat{p}_1 q_2 + p_2 q_1 + q_3, \tag{6.3}$$

*and*

$$q_1 q_3 \mid p_1 q_3 + \widehat{p}_3 q_1 + q_2. \tag{6.4}$$

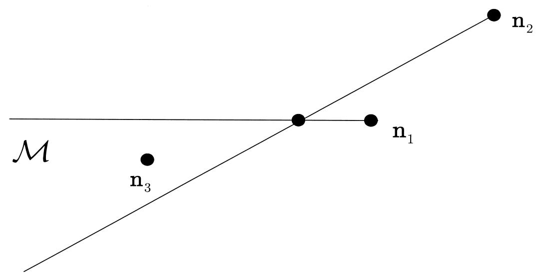



FIGURE 2.

*Proof.* We use Lemma 2.2. Since $\sigma_2$ is a $(p_2, q_2)$-cone and $\sigma_3$ is a $(p_3, q_3)$-cone, setting $\mathbf{n}_3 = \binom{x}{y}$, we have

$$\left.\begin{array}{r} \left|\det\left(\begin{smallmatrix} x & p_1 \\ y & q_1 \end{smallmatrix}\right)\right| = q_2, \\ \\ \binom{x}{y} \in \mathcal{M} \end{array}\right\} \Longrightarrow q_1 x - p_1 y = -q_2. \tag{6.5}$$

on the one hand, and

$$\left.\begin{array}{r} \left|\det\left(\begin{smallmatrix} x & 1 \\ y & 0 \end{smallmatrix}\right)\right| = q_3, \\ \\ \binom{x}{y} \in \mathcal{M} \end{array}\right\} \Longrightarrow y = -q_3,$$

on the other. Hence, (6.5) gives $x = -\frac{1}{q_1}(q_2 + p_1 q_3)$. Moreover, by the definition of $\widehat{p}_1$ there exists an integer $\lambda$ such that

$$\widehat{p}_1 p_1 - \lambda q_1 = 1.$$

This means that

$$\widehat{p}_1\left(-\tfrac{1}{q_1}(q_2 + p_1 q_3)\right) - \lambda(-q_3) \equiv p_2 (\mathrm{mod}\ q_2),$$

i.e., there is a $\mu \in \mathbb{Z}$ with $\mu q_2 = p_2 + \frac{1}{q_1}\left(\widehat{p}_1(q_2 + p_1 q_3) - \lambda q_3 q_1\right)$. Consequently,

$$\mu q_1 q_2 = \widehat{p}_1 q_2 + q_3(\widehat{p}_1 p_1 - \lambda q_1) + p_2 q_1 = \widehat{p}_1 q_2 + p_2 q_1 + q_3,$$

$\mu \in \mathbb{N}$, and the divisibility condition (6.3) is true. Next, by Lemma 2.2 there is a matrix $\left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ such that $\left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{smallmatrix}\right)\binom{x}{y} = \binom{1}{0}$ and $\left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{smallmatrix}\right)\binom{1}{0} = \binom{p_3}{q_3}$, i.e., $\mathfrak{a} = p_3$, $\mathfrak{c} = q_3$, and

$$\left\{\begin{array}{l} q_3 x + \mathfrak{d} y = q_3 x - \mathfrak{d} q_3 = 0 \Longrightarrow \mathfrak{d} = x, \\ \left.\begin{array}{c} p_3 x + \mathfrak{b} y = p_3 x - \mathfrak{b} q_3 = 1 \\ x < 0 \end{array}\right\} \Longrightarrow x = \widehat{p}_3 - \kappa q_3, \text{ for some } \kappa \in \mathbb{N}. \end{array}\right.$$

By (6.5),

$$q_1 x - p_1 y = q_1\left(\widehat{p}_3 - \kappa q_3\right) + p_1 q_3 = -q_2 \Longrightarrow \kappa q_1 q_3 = p_1 q_3 + \widehat{p}_3 q_1 + q_2,$$

leading to the divisibility condition (6.4). $\square$

The converse is also true.

**Lemma 6.3.** *Given a triple of pairs $\{(p_i, q_i) | 1 \le i \le 3\}$ of non-negative integers with $p_i < q_i$ and $\gcd(p_i, q_i) = 1$ for $i \in \{1,2,3\}$, and such that*

$$q_1q_2 \mid \widehat{p}_1q_2 + p_2q_1 + q_3 \quad \textit{and} \quad q_1q_3 \mid p_1q_3 + \widehat{p}_3q_1 + q_2,$$

*the 2-dimensional cones*

$$\sigma_1 = \mathbb{R}_{\ge 0}\binom{1}{0} + \mathbb{R}_{\ge 0}\binom{p_1}{q_1}, \quad \sigma_2 = \mathbb{R}_{\ge 0}\binom{p_1}{q_1} + \mathbb{R}_{\ge 0}\binom{-(q_2+p_1q_3)/q_1}{-q_3},$$
$$\textit{and} \;\; \sigma_3 = \mathbb{R}_{\ge 0}\binom{-(q_2+p_1q_3)/q_1}{-q_3} + \mathbb{R}_{\ge 0}\binom{1}{0},$$

*(written by means of their minimal generators) compose, together with their faces, a complete fan in $\mathbb{R}^2$ and $\sigma_i$ is a $(p_i, q_i)$-cone, for $i \in \{1,2,3\}$.*

*Proof.* Obviously, $\sigma_1$ is a $(p_1, q_1)$-cone and

$$\det\begin{pmatrix} p_1 & -(q_2+p_1q_3)/q_1 \\ q_1 & -q_3 \end{pmatrix} = q_2, \quad \det\begin{pmatrix} -(q_2+p_1q_3)/q_1 & 1 \\ -q_3 & 0 \end{pmatrix} = q_3.$$

Furthermore,

$$q_1q_3 \mid p_1q_3 + \widehat{p}_3q_1 + q_2 \Longrightarrow q_1 \mid q_2 + p_1q_3 \Longrightarrow \binom{-(q_2+p_1q_3)/q_1}{-q_3} \in \mathbb{Z}^2,$$

and setting $\delta := \gcd(q_2 + p_1q_3, q_1q_3)$ we obtain

$$\delta \mid p_1q_3 + \widehat{p}_3q_1 + q_2 \Longrightarrow \delta \mid \widehat{p}_3q_1 \Longrightarrow \delta \mid \widehat{p}_3p_3q_1.$$

Since there exists an integer $\gamma$ with $\widehat{p}_3p_3 - \gamma q_3 = 1$, we have

$$\left.\begin{array}{r} \delta \mid (\gamma q_3 + 1)\, q_1 \\ \delta \mid q_3q_1 \Longrightarrow \delta \mid \gamma q_3q_1 \end{array}\right\} \Longrightarrow \delta \mid q_1.$$

This divisibility condition is equivalent to: $\gcd(\frac{1}{q_1}(q_2+p_1q_3), q_3) = 1$, and therefore $\binom{-(q_2+p_1q_3)/q_1}{-q_3}$ is primitive. On the other hand,

$$q_1q_2 \mid \widehat{p}_1q_2 + p_2q_1 + q_3 \Longrightarrow \exists\, \mu \in \mathbb{N} : \mu q_1q_2 = \widehat{p}_1q_2 + p_2q_1 + q_3.$$

Since there exists an integer $\lambda$ with $\widehat{p}_1p_1 - \lambda q_1 = 1$, and

$$\mu q_1q_2 = \widehat{p}_1q_2 + q_3(\widehat{p}_1p_1 - \lambda q_1) + p_2q_1 \Longrightarrow \widehat{p}_1(-\tfrac{1}{q_1}(q_2+p_1q_3)) - \lambda(-q_3) \equiv p_2 (\mathrm{mod}\ q_2),$$

$\sigma_2$ is a $(p_2, q_2)$-cone. Finally,

$$q_1q_3 \mid p_1q_3 + \widehat{p}_3q_1 + q_2 \Longrightarrow \exists\, \kappa \in \mathbb{N} : \kappa q_1q_3 = p_1q_3 + \widehat{p}_3q_1 + q_2, \text{ i.e.,}$$

$$q_1(\widehat{p}_3 - \kappa q_3) + p_1q_3 = -q_2 = q_1(-\tfrac{1}{q_1}(q_2+p_1q_3)) + p_1q_3 \Longrightarrow -\tfrac{1}{q_1}(q_2+p_1q_3) = \widehat{p}_3 - \kappa q_3,$$

giving

$$\begin{pmatrix} p_3 & \frac{1}{q_3}(p_3\widehat{p}_3 - 1) - \kappa p_3 \\ q_3 & \widehat{p}_3 - \kappa q_3 \end{pmatrix} \begin{pmatrix} -\frac{1}{q_1}(q_2+p_1q_3) \\ -q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} p_3 & \frac{1}{q_3}(p_3\widehat{p}_3 - 1) - \kappa p_3 \\ q_3 & \widehat{p}_3 - \kappa q_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_3 \\ q_3 \end{pmatrix}.$$

Hence, as it is explained in the proof of Proposition 2.4, the cone $\sigma_3$ has to be a $(p_3, q_3)$-cone. □

**Lemma 6.4.** *Every compact toric surface $X_\Delta$ having Picard number $\rho(X_\Delta) = 1$ is a log Del Pezzo surface.*

*Proof.* If $X_\Delta$ is a compact toric surface with $\rho(X_\Delta) = 1$, then the minimal generators $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ of the tree cones (6.1) of $\Delta$ have to be in general position because the cones are strongly convex. Hence, $\mathrm{conv}(\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\})$ has to be an LDP-triangle. □

**Note 6.5.** Compact toric surfaces $X_\Delta$ having Picard number $\rho(X_\Delta) \geq 2$ are not always log Del Pezzo surfaces. For instance, the smooth compact surfaces $X_\Delta$ with $\rho(X_\Delta) = 2$ are the Hirzebruch surfaces $\mathbb{F}_\kappa$, $\kappa \geq 0$ (cf. [13, Corollary 1.29, p. 45]); among them, only $\mathbb{F}_0 \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{F}_1$ (i.e., a $\mathbb{P}^2_{\mathbb{C}}$ blown up at one point) are Del Pezzo surfaces (see [13, Proposition 2.21, p. 88] or [7, Theorem V.8.2, p. 192]). The geometric reason for that is actually very simple: Since $\mathbb{F}_\kappa$ can be viewed as the toric surface associated to the fan having $\binom{0}{1}, \binom{1}{0}, \binom{0}{-1}$ and $\binom{-1}{\kappa}$ as minimal generators of its rays, setting $\mathbf{T} := \mathrm{conv}\left(\left\{\binom{1}{0}, \binom{0}{-1}, \binom{-1}{\kappa}\right\}\right)$ we see that $\binom{0}{1} \in \partial\,\mathbf{T}$ for $\kappa = 2$, and $\binom{0}{1} \in \mathrm{int}(\mathbf{T})$ for $\kappa \geq 3$.

## 7. Classification strategy for $\rho\,(X_\Delta) = 1$ and $\ell = 3$

**Definition 7.1.** We call a triple of pairs

$$\left\{ (p_i, q_i) \in \mathbb{Z}^2 \,\middle|\, 1 \leq i \leq 3 \right\}, \ 0 \leq p_i < q_i, \text{ with } \gcd(p_i, q_i) = 1, \forall i \in \{1,2,3\}, \tag{7.1}$$

*admissible* whenever it satisfies both divisibility conditions

$$\boxed{q_1 q_2 \mid \widehat{p}_1 q_2 + p_2 q_1 + q_3} \tag{7.2}$$

and

$$\boxed{q_1 q_3 \mid p_1 q_3 + \widehat{p}_3 q_1 + q_2} \tag{7.3}$$

To classify all toric log Del Pezzo surfaces $X_\Delta$ having Picard number 1 and index $\ell = 3$ *up to isomorphism* it suffices (by Lemmas 6.2, 6.3, and 6.4, and Theorem 4.4) to determine all admissible triples of pairs, and consequently the fans $\Delta$ having

$$\sigma_1 = \mathbb{R}_{\geq 0}\mathbf{n}_1 + \mathbb{R}_{\geq 0}\mathbf{n}_2, \quad \sigma_2 = \mathbb{R}_{\geq 0}\mathbf{n}_2 + \mathbb{R}_{\geq 0}\mathbf{n}_3, \quad \sigma_3 = \mathbb{R}_{\geq 0}\mathbf{n}_3 + \mathbb{R}_{\geq 0}\mathbf{n}_1,$$

as 2-dimensional cones, with $\mathbf{n}_1 = \binom{1}{0}$, $\mathbf{n}_2 = \binom{p_1}{q_1}$, $\mathbf{n}_3 = \binom{-(q_2+p_1q_3)/q_1}{-q_3}$ as minimal generators, and $Q_\Delta = \mathrm{conv}(\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\})$ as their LDP-polygons, so that

$$l_i = l_{\sigma_i} \in \{1,3\}, \forall i \in \{1,2,3\}, \text{ and } \ l_k = 3 \ \text{ for at least one } \ k \in \{1,2,3\}, \tag{7.4}$$

(see (5.1)). From now on we may assume w.l.o.g that $l_1 = 3$. We also keep in mind the two auxiliary conditions

$$\boxed{s_1 + s_2 + s_3 \leq -\sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} K(E^{(i)})^2 + 8} \tag{7.5}$$

(where, for our convenience, we set $s_i := 0$ for $i \in J_\Delta$, cf. (4.3)), and

$$\boxed{\sum_{i \in \breve{I}_\Delta} q_i \leq \tfrac{1}{3} \sum_{i \in I_\Delta \smallsetminus \breve{I}_\Delta} q_i + \sharp(I_\Delta) + 5} \tag{7.6}$$

(following from (5.2) and (5.3), respectively, for $\nu = \ell = 3$) which have to be satisfied because of Lemma 6.4. By assumption, each pair $(p_i, q_i)$ (belonging to a triple (7.1) which will be considered as "candidate" for being admissible) is necessarily of a specific *type*. All possible types are determined by conditions (7.4), (3.4) and (3.5), and are listed in Table 1. (Since $l_1 = 3$, $(p_1, q_1)$ can be of type $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ or $\mathbf{5}$.)

| **Types** | $p_i$ | $\widehat{p}_i$ | $q_i$ | $s_i$ | $-K(E^{(i)})^2$ |
|---|---|---|---|---|---|
| **1** | 2 | 2 | 3 | 1 | $\frac{1}{3}$ |
| **2** | $3\xi_i+2$ | $p_i$ | $9\xi_i+3$ | $\xi_i+2$ | $\frac{4}{3}$ |
| **3** | $3\xi_i+1$ | $2p_i-1$ $(=6\xi_i+1)$ | $9\xi_i$ | $\xi_i+1$ | 2 |
| **4** | $6\xi_i+5$ | $p_i$ | $9\xi_i+6$ | $\xi_i+1$ | $\frac{8}{3}$ |
| **5** | $6\xi_i+1$ | $\frac{1}{2}(p_i+1)$ $(=3\xi_i+1)$ | $9\xi_i$ | $\xi_i+1$ | 2 |
| **6** | 1 | 1 | $\geq 2$ | $q_i-1$ | 0 |
| **7** | 0 | 0 | 1 | 0 | —— |

Table 1.

Here, $\xi_i$ denotes an integer which is *positive* for types **2**, **3** and **5**, and *non-negative* for type **4**. (In particular, the entries of the last two columns are computed by the continued fraction expansions mentioned in Note 3.3 and by the formula (3.3).) Although the pairs $(p_i, q_i)$ of type **2** (resp., of type **3**, **4**, **5** or **6**) are *infinitely many*, conditions (7.2), (7.3), (7.5) and (7.6) force the testable triples of pairs (7.1) to be admissible only in *finitely many* cases.

**Note 7.2.** If $\mathrm{orb}(\sigma_2)$ is a non-Gorenstein singularity, then (7.2) implies

$$\boxed{[\widehat{p}_1 q_2 + p_2 q_1 + q_3]_9 = 0} \tag{7.7}$$

(where $[t]_9$ denotes the remainder in the division of a $t \in \mathbb{Z}$ by 9) because $3 \mid q_1$ and $3 \mid q_2$. Analogously, if $\mathrm{orb}(\sigma_3)$ is a non-Gorenstein singularity, then (7.3) implies

$$\boxed{[p_1 q_3 + \widehat{p}_3 q_1 + q_2]_9 = 0} \tag{7.8}$$

These weaker, necessary conditions (7.7) and (7.8) turn out to be very useful in proving that several triples of pairs (7.1) are not admissible.

The proof of Theorem 1.3 will follow in four steps:

► <u>**Step 1**</u>: We determine which of the triples of pairs (7.1) corresponding to the $125\ (=5^3)$ possible type combinations $(\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$, are admissible, i.e., those $X_\Delta$'s with exactly *three* non-Gorenstein singularities.

► <u>**Step 2**</u>: We determine which of the triples of pairs (7.1) corresponding to the $100\ (=2\cdot(5^2\cdot 2))$ type combinations $(\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_1 \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$ and

$$(\alpha_2, \alpha_3) \in (\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\} \times \{\mathbf{6},\mathbf{7}\}) \cup (\{\mathbf{6},\mathbf{7}\} \times \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}),$$

are admissible, i.e., those $X_\Delta$'s with exactly *two* non-Gorenstein singularities.

► <u>**Step 3**</u>: We do the same for the triples of pairs (7.1) corresponding to the 20 type combinations $(\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_1 \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$ and $\alpha_2, \alpha_3 \in \{\mathbf{6},\mathbf{7}\}$, i.e., for those $X_\Delta$'s with exactly *one* non-Gorenstein singularity.

► <u>**Step 4**</u>: We find out the WVE$^2$C-graphs $\mathfrak{G}_\Delta$ for those $X_\Delta$'s determined in steps 1-3, and then, using Theorem 4.4, we pick out a suitable, minimal set of representatives of $X_\Delta$'s all of whose members are pairwise non-isomorphic. Finally, we identify the chosen $X_\Delta$'s with weighted projective planes or quotients thereof by applying Lemma 6.1.

## 8. Proof of Theorem 1.3: Step 1

**Lemma 8.1.** *Among the* 125 *possible combinations* $(\alpha_1, \alpha_2, \alpha_3)$ *of types of triples of pairs* (7.1), *with* $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, *there are only* 32 *satisfying simultaneously conditions* (7.7) *and* (7.8); *namely,*

$$(\mathbf{1}, \mathbf{3}, \mathbf{4}), (\mathbf{1}, \mathbf{4}, \mathbf{5}), (\mathbf{2}, \mathbf{3}, \mathbf{4}), (\mathbf{2}, \mathbf{4}, \mathbf{5}), (\mathbf{3}, \mathbf{3}, \mathbf{3}), (\mathbf{3}, \mathbf{3}, \mathbf{5}), (\mathbf{3}, \mathbf{5}, \mathbf{5}), (\mathbf{5}, \mathbf{5}, \mathbf{5}),$$

*together with their permutations.*

| **Case** | $[\widehat{p}_1]_9$ | $[q_1]_9$ | $[p_2]_9$ | $[q_2]_9$ | $[\widehat{p}_1 q_2 + p_2 q_1]_9$ | (7.7) is true only if |
|---|---|---|---|---|---|---|
| $(\mathbf{1}, \mathbf{1}, \alpha_3)$ | 2 | 3 | 2 | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{1}, \mathbf{2}, \alpha_3)$ | 2 | 3 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{1}, \mathbf{3}, \alpha_3)$ | 2 | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{1}, \mathbf{4}, \alpha_3)$ | 2 | 3 | $\in \{2,5,8\}$ | 6 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{1}, \mathbf{5}, \alpha_3)$ | 2 | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{2}, \mathbf{1}, \alpha_3)$ | $\in \{2,5,8\}$ | 3 | 2 | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{2}, \mathbf{2}, \alpha_3)$ | $\in \{2,5,8\}$ | 3 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{2}, \mathbf{3}, \alpha_3)$ | $\in \{2,5,8\}$ | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{2}, \mathbf{4}, \alpha_3)$ | $\in \{2,5,8\}$ | 3 | $\in \{2,5,8\}$ | 6 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{2}, \mathbf{5}, \alpha_3)$ | $\in \{2,5,8\}$ | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{3}, \mathbf{1}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | 2 | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{3}, \mathbf{2}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{3}, \mathbf{3}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{3}, \mathbf{4}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_3 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{3}, \mathbf{5}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \mathbf{1}, \alpha_3)$ | $\in \{2,5,8\}$ | 6 | 2 | 3 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \mathbf{2}, \alpha_3)$ | $\in \{2,5,8\}$ | 6 | $\in \{2,5,8\}$ | 3 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \mathbf{3}, \alpha_3)$ | $\in \{2,5,8\}$ | 6 | $\in \{1,4,7\}$ | 0 | 6 | $\alpha_3 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{4}, \mathbf{4}, \alpha_3)$ | $\in \{2,5,8\}$ | 6 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_3 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{4}, \mathbf{5}, \alpha_3)$ | $\in \{2,5,8\}$ | 6 | $\in \{1,4,7\}$ | 0 | 6 | $\alpha_3 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{5}, \mathbf{1}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | 2 | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{5}, \mathbf{2}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_3 = \mathbf{4}$ |
| $(\mathbf{5}, \mathbf{3}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{5}, \mathbf{4}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_3 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{5}, \mathbf{5}, \alpha_3)$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_3 \in \{\mathbf{3}, \mathbf{5}\}$ |

Table 2.

| **Case** | $[p_1]_9$ | $[q_1]_9$ | $[\widehat{p}_3]_9$ | $[q_3]_9$ | $[p_1 q_3 + \widehat{p}_3 q_1]_9$ | (7.8) is true only if |
|---|---|---|---|---|---|---|
| $(\mathbf{1}, \alpha_2, \mathbf{1})$ | 2 | 3 | 2 | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{1}, \alpha_2, \mathbf{2})$ | 2 | 3 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{1}, \alpha_2, \mathbf{3})$ | 2 | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{1}, \alpha_2, \mathbf{4})$ | 2 | 3 | $\in \{2,5,8\}$ | 6 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{1}, \alpha_2, \mathbf{5})$ | 2 | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{2}, \alpha_2, \mathbf{1})$ | $\in \{2,5,8\}$ | 3 | 2 | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{2}, \alpha_2, \mathbf{2})$ | $\in \{2,5,8\}$ | 3 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{2}, \alpha_2, \mathbf{3})$ | $\in \{2,5,8\}$ | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{2}, \alpha_2, \mathbf{4})$ | $\in \{2,5,8\}$ | 3 | $\in \{2,5,8\}$ | 6 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{2}, \alpha_2, \mathbf{5})$ | $\in \{2,5,8\}$ | 3 | $\in \{1,4,7\}$ | 0 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{3}, \alpha_2, \mathbf{1})$ | $\in \{1,4,7\}$ | 0 | 2 | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{3}, \alpha_2, \mathbf{2})$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{3}, \alpha_2, \mathbf{3})$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{3}, \alpha_2, \mathbf{4})$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_2 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{3}, \alpha_2, \mathbf{5})$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \alpha_2, \mathbf{1})$ | $\in \{2,5,8\}$ | 6 | 2 | 3 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \alpha_2, \mathbf{2})$ | $\in \{2,5,8\}$ | 6 | $\in \{2,5,8\}$ | 3 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{4}, \alpha_2, \mathbf{3})$ | $\in \{2,5,8\}$ | 6 | $\in \{1,4,7\}$ | 0 | 6 | $\alpha_2 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{4}, \alpha_2, \mathbf{4})$ | $\in \{2,5,8\}$ | 6 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_2 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{4}, \alpha_2, \mathbf{5})$ | $\in \{2,5,8\}$ | 6 | $\in \{1,4,7\}$ | 0 | 6 | $\alpha_2 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{5}, \alpha_2, \mathbf{1})$ | $\in \{1,4,7\}$ | 0 | 2 | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{5}, \alpha_2, \mathbf{2})$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 3 | 3 | $\alpha_2 = \mathbf{4}$ |
| $(\mathbf{5}, \alpha_2, \mathbf{3})$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |
| $(\mathbf{5}, \alpha_2, \mathbf{4})$ | $\in \{1,4,7\}$ | 0 | $\in \{2,5,8\}$ | 6 | 6 | $\alpha_2 \in \{\mathbf{1}, \mathbf{2}\}$ |
| $(\mathbf{5}, \alpha_2, \mathbf{5})$ | $\in \{1,4,7\}$ | 0 | $\in \{1,4,7\}$ | 0 | 0 | $\alpha_2 \in \{\mathbf{3}, \mathbf{5}\}$ |

Table 3.

*Proof.* By Table 2 there are 38 combinations $(\alpha_1, \alpha_2, \alpha_3)$ of types of triples (7.1), with $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, satisfying condition (7.7). Correspondingly, Table 3 shows that there are 38 combinations $(\alpha_1, \alpha_2, \alpha_3)$ of types of triples (7.1), with $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, satisfying condition (7.8). Obviously, the combinations $(\alpha_1, \alpha_2, \alpha_3)$ of types of triples of pairs (7.1), with $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, satisfying *both* (7.7) and (7.8), are the 32 combinations given in the statement of Lemma. □

**Lemma 8.2.** *There are no admissible triples of pairs* (7.1) *among those corresponding to the* 125 *type combinations* $(\alpha_1, \alpha_2, \alpha_3)$ *with* $\alpha_1, \alpha_2, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

*Sketch of proof.* First, we express the triples of pairs $\left\{ (p_i, q_i) \in \mathbb{Z}^2 \,\middle|\, 1 \le i \le 3 \right\}$ corresponding to the 32 type combinations $(\alpha_1, \alpha_2, \alpha_3)$ found in Lemma 8.1 in terms of $\xi_i$ for $i \in \{1,2,3\}$ as in Table 1. Setting

$$\mathfrak{A}_j := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{Z}^3 \,\middle|\, \begin{array}{c} \xi_1 + \xi_2 + \xi_3 \le 10,\ \xi_j \ge 0, \\ \text{and } \xi_k \ge 1,\ \forall k \in \{1,2,3\} \smallsetminus \{j\} \end{array} \right\},$$

for $j \in \{1,2,3\}$,

$$\mathfrak{A}_{j,k} := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{Z}^3 \,\middle|\, \begin{array}{c} \xi_1 + \xi_2 + \xi_3 \le 10,\ \xi_j = 0, \xi_k \ge 0, \\ \text{and } \xi_\mu \ge 1,\ \forall \mu \in \{1,2,3\} \smallsetminus \{j,k\} \end{array} \right\},$$

for $j, k \in \{1,2,3\}$, $j \ne k$, and

$$\mathfrak{B} := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{Z}^3 \,\middle|\, \xi_1 + \xi_2 + \xi_3 \le 11,\ \xi_1, \xi_2, \xi_3 \ge 1 \right\},$$

we explain what condition (7.5) means for each of these 32 cases in Table 4.

| **Case** | Condition (7.5) | **Case** | Condition (7.5) |
|---|---|---|---|
| $(\mathbf{1},\mathbf{3},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{1,3}$ | $(\mathbf{4},\mathbf{1},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{2,1}$ |
| $(\mathbf{1},\mathbf{4},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{1,2}$ | $(\mathbf{4},\mathbf{1},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{2,1}$ |
| $(\mathbf{1},\mathbf{4},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{1,2}$ | $(\mathbf{4},\mathbf{2},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_1$ |
| $(\mathbf{1},\mathbf{5},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{1,3}$ | $(\mathbf{4},\mathbf{2},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_1$ |
| $(\mathbf{2},\mathbf{3},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_3$ | $(\mathbf{4},\mathbf{3},\mathbf{1})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{3,1}$ |
| $(\mathbf{2},\mathbf{4},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_2$ | $(\mathbf{4},\mathbf{3},\mathbf{2})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_1$ |
| $(\mathbf{2},\mathbf{4},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_2$ | $(\mathbf{4},\mathbf{5},\mathbf{1})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{3,1}$ |
| $(\mathbf{2},\mathbf{5},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_3$ | $(\mathbf{4},\mathbf{5},\mathbf{2})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_1$ |
| $(\mathbf{3},\mathbf{1},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{2,3}$ | $(\mathbf{5},\mathbf{1},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{2,3}$ |
| $(\mathbf{3},\mathbf{2},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_3$ | $(\mathbf{5},\mathbf{2},\mathbf{4})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_3$ |
| $(\mathbf{3},\mathbf{3},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ | $(\mathbf{5},\mathbf{3},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ |
| $(\mathbf{3},\mathbf{3},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ | $(\mathbf{5},\mathbf{3},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ |
| $(\mathbf{3},\mathbf{4},\mathbf{1})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{3,2}$ | $(\mathbf{5},\mathbf{4},\mathbf{1})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_{3,2}$ |
| $(\mathbf{3},\mathbf{4},\mathbf{2})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_2$ | $(\mathbf{5},\mathbf{4},\mathbf{2})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{A}_2$ |
| $(\mathbf{3},\mathbf{5},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ | $(\mathbf{5},\mathbf{5},\mathbf{3})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ |
| $(\mathbf{3},\mathbf{5},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ | $(\mathbf{5},\mathbf{5},\mathbf{5})$ | $(\xi_1,\xi_2,\xi_3) \in \mathfrak{B}$ |

Table 4.

Note that

$$\sharp(\mathfrak{A}_j) = \sum_{\kappa=2}^{10} \binom{\kappa-1}{1} + \sum_{\kappa=3}^{10} \binom{\kappa-1}{2} = 165,\ \sharp(\mathfrak{A}_{j,k}) = 55,\ \sharp(\mathfrak{B}) = \binom{11}{3} = 165.$$

One can, of course, test directly the validity of (7.2) and (7.3) for all these possibilities. Nevertheless, there is a more economic way to proceed by using reductio ad absurdum. Let us discuss it exemplarily in the case $(\mathbf{2}, \mathbf{3}, \mathbf{4})$ in which

| $p_1$ | $\widehat{p}_1$ | $q_1$ | $p_2$ | $\widehat{p}_2$ | $q_2$ | $p_3$ | $\widehat{p}_3$ | $q_3$ |
|---|---|---|---|---|---|---|---|---|
| $3\xi_1 + 2$ | $3\xi_1 + 2$ | $9\xi_1 + 3$ | $3\xi_2 + 1$ | $6\xi_2 + 1$ | $9\xi_2$ | $6\xi_3 + 5$ | $6\xi_3 + 5$ | $9\xi_3 + 6$ |

for a 3-tuple $(\xi_1, \xi_2, \xi_3) \in \mathfrak{A}_3$. If $\left\{ (p_i, q_i) \in \mathbb{Z}^2 \,\middle|\, 1 \leq i \leq 3 \right\}$ were an admissible triple of pairs, then (7.2) would give

$$(9\xi_1 + 3)(9\xi_2) = 27\xi_2 + 81\xi_1\xi_2 \mid 9\xi_1 + 27\xi_2 + 9\xi_3 + 54\xi_1\xi_2 + 9, \text{ i.e.,}$$

$$3(3\xi_1 + 1)\xi_2 \mid \xi_1 + 3\xi_2 + \xi_3 + 6\xi_1\xi_2 + 1 = 3(3\xi_1 + 1)\xi_2 - 3\xi_1\xi_2 + \xi_1 + \xi_3 + 1, \quad (8.1)$$

meaning that $3(3\xi_1 + 1)\xi_2 \mid 3\xi_1\xi_2 - \xi_1 - \xi_3 - 1$. Therefore, $3\xi_1\xi_2 - \xi_1 - \xi_3 - 1 \leq 0$ (because otherwise we would deduce that $3\xi_2 + 9\xi_1\xi_2 \leq 3\xi_1\xi_2 - \xi_1 - \xi_3 - 1$, i.e., that $10 \leq \xi_1 + 3\xi_2 + \xi_3 + 6\xi_1\xi_2 \leq -1$, a contradiction). Consequently,

$$\begin{aligned} 3\xi_1\xi_2 - 1 \leq \xi_1 + \xi_3 \leq 10 - \xi_2 &\Longrightarrow 4 \leq (3\xi_1 + 1)\xi_2 \leq 11 \\ &\Longrightarrow (\xi_1, \xi_2) \in \{(1,1), (1,2), (2,1), (3,1)\}\,. \end{aligned} \quad (8.2)$$

Since $0 \leq \xi_3 \leq 8$, (8.1) and (8.2) would determine the values of $\xi_3$ as follows:

| $(\xi_1, \xi_2, \xi_3)$ | $p_1$ | $q_1$ | $q_2$ | $\widehat{p}_3$ | $q_3$ | $q_1 q_3$ | $p_1 q_3 + \widehat{p}_3 q_1 + q_2$ |
|---|---|---|---|---|---|---|---|
| $(1,1,1)$ | 5 | 12 | 9 | 11 | 15 | 180 | 216 |
| $(1,2,4)$ | 5 | 12 | 18 | 29 | 42 | 504 | 576 |
| $(2,1,3)$ | 8 | 21 | 9 | 23 | 33 | 693 | 756 |
| $(3,1,5)$ | 11 | 30 | 9 | 35 | 51 | 1530 | 1620 |

Hence, these four 3-tuples $(\xi_1, \xi_2, \xi_3) \in \mathfrak{A}_3$ would provide numbers $p_1, q_1, q_2, \widehat{p}_3, q_3$ which do not satisfy (7.3)! Using analogous arguments one shows that none of the remaining 31 cases leads to admissible triples of pairs. □

## 9. Proof of Theorem 1.3: Step 2

**Lemma 9.1.** *There are no admissible triples of pairs* (7.1) *among those corresponding to the type combinations* $(\alpha_1, \alpha_2, \alpha_3)$ *with* $\alpha_1 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ *and*

$$(\alpha_2, \alpha_3) \in (\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\} \times \{\mathbf{7}\}) \cup (\{\mathbf{7}\} \times \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\})\,.$$

*Proof.* If $\alpha_1, \alpha_2 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\alpha_3 = \mathbf{7}$, then $[\widehat{p}_1 q_2 + p_2 q_1]_9 \in \{0, 3, 6\}$ (cf. the sixth column of Table 2) and $q_3 = 1$, i.e., $[\widehat{p}_1 q_2 + p_2 q_1 + q_3]_9 \in \{1, 4, 7\}$. Thus, condition (7.7) is not satisfied. Analogously, one shows that condition (7.8) is not satisfied whenever $\alpha_1, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\alpha_2 = \mathbf{7}$. □

**Lemma 9.2.** *There exist exactly* 10 *admissible triples of pairs* (7.1) *among those corresponding to the type combinations* $(\alpha_1, \alpha_2, \alpha_3)$ *with* $\alpha_1 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ *and*

$$(\alpha_2, \alpha_3) \in (\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\} \times \{\mathbf{6}\}) \cup (\{\mathbf{6}\} \times \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\})\,.$$

*Sketch of proof.* For $\alpha_1, \alpha_2 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\alpha_3 = \mathbf{6}$ we build Table 5. In its second column we tabulate $[\widehat{p}_1 q_2 + p_2 q_1]_9$ (cf. the sixth column of Table 2). After having expressed $q_1$, $q_2$ in terms of $\xi_1, \xi_2$ (as in Table 1) we write the restrictions (inequalities) coming from (7.5) in its third column. The fourth column contains the values of $q_3$ so that both (7.5) and (7.7) are true. (In particular, in the case $(\mathbf{2}, \mathbf{2}, \mathbf{6})$ the expected value $q_3 = 6$ is impossible because $\xi_1, \xi_2 \geq 1$.) Finally, the last column informs us whether (7.8) is true for these $q_3$'s.

| **Case** | $[\widehat{p}_1 q_2 + p_2 q_1]_9$ | (7.5) is true whenever | (7.5) & (7.7) true only if $q_3$ equals | Is (7.8) true for these $q_3$'s? |
|---|---|---|---|---|
| $(\mathbf{1},\mathbf{1},\mathbf{6})$ | 3 | $2 \leq q_3 \leq 7$ | 6 | YES |
| $(\mathbf{1},\mathbf{2},\mathbf{6})$ | 3 | $3 \leq \xi_2 + q_3 \leq 7$ | 6 | YES |
| $(\mathbf{1},\mathbf{3},\mathbf{6})$ | 3 | $3 \leq \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{1},\mathbf{4},\mathbf{6})$ | 0 | $2 \leq \xi_2 + q_3 \leq 10$ | 9 | YES |
| $(\mathbf{1},\mathbf{5},\mathbf{6})$ | 3 | $3 \leq \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{2},\mathbf{1},\mathbf{6})$ | 3 | $3 \leq \xi_1 + q_3 \leq 7$ | 6 | YES |
| $(\mathbf{2},\mathbf{2},\mathbf{6})$ | 3 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 7$ | 6 (impossible) | —— |
| $(\mathbf{2},\mathbf{3},\mathbf{6})$ | 3 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{2},\mathbf{4},\mathbf{6})$ | 0 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 10$ | 9 | YES |
| $(\mathbf{2},\mathbf{5},\mathbf{6})$ | 3 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{3},\mathbf{1},\mathbf{6})$ | 3 | $3 \leq \xi_1 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{3},\mathbf{2},\mathbf{6})$ | 3 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{3},\mathbf{3},\mathbf{6})$ | 0 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 9 | YES |
| $(\mathbf{3},\mathbf{4},\mathbf{6})$ | 6 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 3 | YES |
| $(\mathbf{3},\mathbf{5},\mathbf{6})$ | 0 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 9 | YES |
| $(\mathbf{4},\mathbf{1},\mathbf{6})$ | 0 | $2 \leq \xi_1 + q_3 \leq 10$ | 9 | YES |
| $(\mathbf{4},\mathbf{2},\mathbf{6})$ | 0 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 10$ | 9 | YES |
| $(\mathbf{4},\mathbf{3},\mathbf{6})$ | 6 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 3 | NO |
| $(\mathbf{4},\mathbf{4},\mathbf{6})$ | 6 | $2 \leq \xi_1 + \xi_2 + q_3 \leq 12$ | 3 or 12 | YES |
| $(\mathbf{4},\mathbf{5},\mathbf{6})$ | 6 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 3 | NO |
| $(\mathbf{5},\mathbf{1},\mathbf{6})$ | 3 | $3 \leq \xi_1 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{5},\mathbf{2},\mathbf{6})$ | 3 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 9$ | 6 | NO |
| $(\mathbf{5},\mathbf{3},\mathbf{6})$ | 0 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 9 | YES |
| $(\mathbf{5},\mathbf{4},\mathbf{6})$ | 6 | $3 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 3 | YES |
| $(\mathbf{5},\mathbf{5},\mathbf{6})$ | 0 | $4 \leq \xi_1 + \xi_2 + q_3 \leq 11$ | 9 | YES |

TABLE 5.

Next, we analyze in detail the 14 cases for which the answer is "YES".

• In the case $(\mathbf{1},\mathbf{1},\mathbf{6})$ we have $q_3 = 6$ and we obtain just one admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 2 | 3 | 1 | 6 |

(9.1)

• In cases $(\mathbf{1},\mathbf{2},\mathbf{6})$ and $(\mathbf{2},\mathbf{1},\mathbf{6})$ we have $\xi_2 = 1, q_3 = 6$, and $\xi_1 = 1, q_3 = 6$, respectively, and (7.2) cannot be satisfied (because $36 \nmid 45$). Hence, there are no admissible triples of pairs.

• In cases $(\mathbf{1},\mathbf{4},\mathbf{6})$ and $(\mathbf{4},\mathbf{1},\mathbf{6})$ we have $\xi_2 \in \{0,1\}, q_3 = 9$, and $\xi_1 \in \{0,1\}, q_3 = 9$, respectively, and (7.2) cannot be satisfied for $\xi_2 = 1$, resp. for $\xi_1 = 1$ (because $45 \nmid 72$). For this reason, the only triples of pairs which are admissible (i.e., for which both (7.2) and (7.3) are satified) are

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 1 | 9 |

(9.2)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 2 | 3 | 1 | 9 |

(9.3)

• In cases $(\mathbf{2},\mathbf{4},\mathbf{6})$ and $(\mathbf{4},\mathbf{2},\mathbf{6})$ we have necessarily $\xi_1 = 1$, $\xi_2 = 0$, $q_3 = 9$, and $\xi_1 = 0$, $\xi_2 = 1$, $q_3 = 9$, respectively, and (7.2) cannot be satisfied (because $72 \nmid 99$). Hence, there are no admissible triples of pairs.

• In cases $(\mathbf{3},\mathbf{3},\mathbf{6})$ and $(\mathbf{5},\mathbf{5},\mathbf{6})$ we have necessarily $\xi_1 = \xi_2 = 1$, $q_3 = 9$, and (7.2) cannot be satisfied (because $81 \nmid 108$). Therefore, there are no admissible triples of pairs.

• In cases $(\mathbf{3},\mathbf{4},\mathbf{6})$ and $(\mathbf{5},\mathbf{4},\mathbf{6})$ we have $q_3 = 3$ and $\xi_1 + \xi_2 \in \{1,\ldots,8\}$ with $\xi_1 \geq 1$ and $\xi_2 \geq 0$. If (7.2) were true, then in particular $q_1 \mid \widehat{p}_1 q_2 + q_3$, i.e.,

$$\xi_1 \mid \xi_2 + 1 \Longrightarrow (\xi_1, \xi_2) \in \{\,(1,j) | \, 0 \leq j \leq 7\} \cup \{(2,1),(2,3),(2,5),(3,2),(3,5),(4,3)\}.$$

As for everyone of the 14 possible values of $(\xi_1, \xi_2)$ at least one of the divisibility conditions (7.2) and (7.3) is violated, there are no admissible triples of pairs.

• Case $(\mathbf{3}, \mathbf{5}, \mathbf{6})$: $\xi_1 = \xi_2 = 1$, $q_3 = 9$, and (7.2) cannot be satisfied (because $81 \nmid 135$); no admissible triples of pairs occur.

• Case $(\mathbf{4}, \mathbf{4}, \mathbf{6})$: Here, *either* $\xi_1 = \xi_2 = 0$, $q_3 = 12$, giving the admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 5 | 6 | 1 | 12 |

(9.4)

*or* $q_3 = 3$ and $\xi_1 + \xi_2 \in \{0, 1, \ldots, 9\}$ with $\xi_1, \xi_2 \geq 0$. If in the latter case (7.2) were true, then, in particular, $q_1 \mid \widehat{p}_1 q_2 + q_3$, i.e.,

$$\begin{aligned} 9\xi_1 + 6 \mid (6\xi_1 + 5)(9\xi_2 + 6) + 3 &\Longrightarrow 3\xi_1 + 2 \mid (3\xi_1 + 2)(6\xi_2 + 4) + 3\xi_2 + 3 \\ &\Longrightarrow 3\xi_1 + 2 \mid 3\xi_2 + 3 \Longrightarrow 3\xi_1 + 2 \mid \xi_2 + 1, \text{ i.e.,} \end{aligned}$$

$$(\xi_1, \xi_2) \in \{(0,1), (0,3), (0,5), (0,7), (0,9), (1,4), (2,7)\}.$$

As for everyone of the 7 possible values of $(\xi_1, \xi_2)$ at least one of the divisibility conditions (7.2) and (7.3) is violated, there are no further admissible triples of pairs.

• Case $(\mathbf{5}, \mathbf{3}, \mathbf{6})$: $\xi_1 = \xi_2 = 1$, $q_3 = 9$, and we obtain just one admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 7 | 9 | 4 | 9 | 1 | 9 |

(9.5)

Working symmetrically with type combinations $(\alpha_1, \alpha_2, \alpha_3)$, where

$$\alpha_1, \alpha_3 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\} \text{ and } \alpha_2 = \mathbf{6},$$

we determine the admissible triples of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 6 | 2 | 3 |

(9.6)

in the case $(\mathbf{1}, \mathbf{6}, \mathbf{1})$,

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 9 | 5 | 6 |

(9.7)

in the case $(\mathbf{1}, \mathbf{6}, \mathbf{4})$,

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 4 | 9 | 1 | 9 | 7 | 9 |

(9.8)

in the case $(\mathbf{3}, \mathbf{6}, \mathbf{5})$

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 1 | 9 | 2 | 3 |

(9.9)

in the case $(\mathbf{4}, \mathbf{6}, \mathbf{1})$, and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 1 | 12 | 5 | 6 |

(9.10)

in the case $(\mathbf{4}, \mathbf{6}, \mathbf{4})$. □

## 10. Proof of Theorem 1.3: Step 3

**Lemma 10.1.** *There exist exactly* 23 *admissible triples of pairs* (7.1) *among those corresponding to the type combinations* $(\alpha_1, \alpha_2, \alpha_3)$ *with* $\alpha_1 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ *and* $\alpha_2, \alpha_3 \in \{\mathbf{6}, \mathbf{7}\}$.

*Proof.* For every $\alpha_1 \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ we consider the combinations

| **Case** | $p_2$ | $q_2$ | $p_3 = \widehat{p}_3$ | $q_3$ |
|---|---|---|---|---|
| $(\alpha_1, \mathbf{6}, \mathbf{6})$ | 1 | $\geq 2$ | 1 | $\geq 2$ |
| $(\alpha_1, \mathbf{6}, \mathbf{7})$ | 1 | $\geq 2$ | 0 | 1 |
| $(\alpha_1, \mathbf{7}, \mathbf{6})$ | 0 | 1 | 1 | $\geq 2$ |
| $(\alpha_1, \mathbf{7}, \mathbf{7})$ | 0 | 1 | 0 | 1 |

and examine what happens in each of the twenty cases separately.

• Case $(\mathbf{1}, \mathbf{6}, \mathbf{6})$: Here, and for the next three cases, $p_1 = \widehat{p}_1 = 2$, $q_1 = 3$ and $s_1 = 1$. By (7.5) and (7.6) the pair $(q_2, q_3)$ has to be chosen from the 21 elements of the set

$$\left\{ (q_2, q_3) \in \mathbb{Z}^2 \middle| \, q_2 \geq 2, \;\; q_3 \geq 2, \;\text{ and } q_2 + q_3 \leq 9 \right\}.$$

Taking into account the divisibility conditions (7.2), (7.3), i.e., $3q_2 \mid 2q_2 + q_3 + 3$ and $3q_3 \mid 2q_3 + q_2 + 3$, we obtain $(q_2, q_3) \in \{(2, 5), (5, 2)\}$. Hence, there are two admissible triples of pairs, namely

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 2 | 1 | 5 |

(10.1)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 5 | 1 | 2 |

(10.2)

• Case $(\mathbf{1}, \mathbf{6}, \mathbf{7})$: By (7.5) (or (7.6)) we have $q_2 \leq 8$. By (7.3), $3 \mid q_2 - 1$, i.e., $q_2 \in \{4, 7\}$. The value $q_2 = 7$ does not satisfy (7.2): $3q_2 \mid 2q_2 + 4$. Hence, there is only one admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 1 | 4 | 0 | 1 |

(10.3)

• Case $(\mathbf{1}, \mathbf{7}, \mathbf{6})$: Analogously, we find just one admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 0 | 1 | 1 | 4 |

(10.4)

• Case $(\mathbf{1}, \mathbf{7}, \mathbf{7})$: In this case both divisibility conditions (7.2) and (7.3) are satisfied automatically and lead to the admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 2 | 3 | 0 | 1 | 0 | 1 |

• Case $(\mathbf{2}, \mathbf{6}, \mathbf{6})$: Here, and for the next three cases, $p_1 = \widehat{p}_1 = 3\xi_1 + 2$, $q_1 = 9\xi_1 + 3$ and $s_1 = \xi_1 + 2$ for an integer $\xi_1 \geq 1$. By (7.5) we have $\xi_1 + q_2 + q_3 \leq 9$. Condition (7.2) reads as

$$3(3\xi_1 + 1)q_2 \mid (3\xi_1 + 2)q_2 + (9\xi_1 + 3) + q_3 = 3(3\xi_1 + 1)q_2 - 6\xi_1 q_2 - q_2 + (9\xi_1 + 3) + q_3,$$

i.e.,

$$3(3\xi_1 + 1)q_2 \mid 6\xi_1 q_2 + q_2 - (9\xi_1 + 3) - q_3 \;\;\text{ with }\;\; 6\xi_1 q_2 + q_2 - (9\xi_1 + 3) - q_3 \leq 0.$$

Since $1 \leq \xi_1 \leq 5$,

$$(6\xi_1+1)q_2 \leq 9\xi_1+3+q_3 \leq 9\xi_1+3+(9-\xi_1-q_2) \Longrightarrow (6\xi_1+2)q_2 \leq 8\xi_1+12 \leq 52,$$

implying

$$8 \leq (3\xi_1+1)q_2 \leq 26.$$

These inequalities are satisfied if and only if

$$(\xi_1, q_2) \in \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(3,2)\}.$$

Since $2 \leq q_3 \leq 9-(\xi_1+q_2)$, the divisibility condition (7.2) is true only for $\xi_1 = 1$, $q_2 = 3$ and $q_3 = 2$. (For these values (7.3) is also true.) Hence, the only admissible triple of pairs is the following:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 12 | 1 | 2 | 1 | 2 |

• Case $(\mathbf{2},\mathbf{6},\mathbf{7})$: By (7.5) we have $\xi_1 + q_2 \leq 8$. Condition (7.3) gives

$$3(3\xi_1+1) \mid 3\xi_1+2+q_2 \Longrightarrow 3 \mid q_2-1 \text{ and } 3\xi_1+1 \mid q_2+1.$$

But this means that $(\xi_1, q_2) \in \{(1,7),(2,6)\}$. $(2,6)$ is not permitted because $21 \nmid 14$ and $(1,7)$ violates (7.2), so there are no admissible triples of pairs.

• Case $(\mathbf{2},\mathbf{7},\mathbf{6})$: As in the case $(\mathbf{2},\mathbf{6},\mathbf{7})$ one shows that there are no admissible triples of pairs.

• Case $(\mathbf{2},\mathbf{7},\mathbf{7})$: Conditions (7.2) and (7.3) give $q_1 = 3(p_1-1) \mid p_1+1$, i.e., $p_1 = 2$, but in this case $p_1 \geq 5$. Hence, there are no admissible triples of pairs.

• Case $(\mathbf{3},\mathbf{6},\mathbf{6})$: Here, and for the next three cases, $p_1 = 3\xi_1+1$, $\widehat{p}_1 = 6\xi_1+1$, $q_1 = 9\xi_1$ and $s_1 = \xi_1+1$ for an integer $\xi_1 \geq 1$. By (7.5) we have $\xi_1+q_2+q_3 \leq 11$. Condition (7.2) reads as

$$9\xi_1 q_2 \mid (6\xi_1+1)q_2+9\xi_1+q_3 = 9\xi_1 q_2 - 3\xi_1 q_2 + q_2 + 9\xi_1 + q_3,$$

i.e.,

$$9\xi_1 q_2 \mid 3\xi_1 q_2 - q_2 - 9\xi_1 - q_3 \quad \text{with} \quad 3\xi_1 q_2 - q_2 - 9\xi_1 - q_3 \leq 0.$$

Since $1 \leq \xi_1 \leq 7$,

$$3\xi_1 q_2 \leq q_2+q_3+9\xi_1 \leq 11+8\xi_1 \leq 67 \Longrightarrow 2 \leq \xi_1 q_2 \leq 22.$$

Since $2 \leq q_3 \leq 11-(\xi_1+q_2)$, the divisibility conditions (7.2) and (7.3) are true only for $\xi_1 = 1$, $q_2 = 6$, $q_3 = 3$, or $\xi_1 = 2$, $q_2 = 4$, $q_3 = 2$, leading to two admissible triple of pairs, namely

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 4 | 9 | 1 | 6 | 1 | 3 |

(10.5)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 7 | 18 | 1 | 4 | 1 | 2 |

(10.6)

• Case $(\mathbf{3},\mathbf{6},\mathbf{7})$: By (7.5) we have $\xi_1 + q_2 \leq 10$. Condition (7.3) gives

$$9\xi_1 \mid 3\xi_1+1+q_2 \Longrightarrow 9\xi_1 \mid 6\xi_1-q_2-1, \text{ with } 6\xi_1-q_2-1 \leq 0.$$

Thus,

$$6\xi_1 \leq q_2+1 \leq 11-\xi_1 \Longrightarrow \xi_1 \leq \frac{11}{7} \Longrightarrow \xi_1 = 1.$$

Since $2 \leq q_2 \leq 9$, condition (7.2) (i.e., $9q_2 \mid 7q_2 + 10$) implies $q_2 = 5$. The corresponding admissible triple of pairs is the following:

$$\begin{array}{|c|c|c|c|c|c|}\hline p_1 & q_1 & p_2 & q_2 & p_3 & q_3 \\ \hline 4 & 9 & 1 & 5 & 0 & 1 \\ \hline\end{array} \tag{10.7}$$

• Case $(\mathbf{3},\mathbf{7},\mathbf{6})$: By (7.5) we have $\xi_1 + q_3 \leq 10$. Condition (7.2) gives

$$\xi_1 \mid 6\xi_1 + 1 + q_3 \Longrightarrow 9\xi_1 \mid 3\xi_1 - q_3 - 1, \text{ with } 3\xi_1 - q_3 - 1 \leq 0.$$

Thus,

$$3\xi_1 \leq q_3 + 1 \leq 11 - \xi_1 \Longrightarrow \xi_1 \leq \frac{11}{4} \Longrightarrow \xi_1 \in \{1, 2\}.$$

Since $2 \leq q_3 \leq 9$, condition (7.2) implies $(\xi_1, q_3) \in \{(1,2), (2,5)\}$. $(2,5)$ is not permitted because it violates (7.3). For this reason, the only admissible triple of pairs is the following:

$$\begin{array}{|c|c|c|c|c|c|}\hline p_1 & q_1 & p_2 & q_2 & p_3 & q_3 \\ \hline 4 & 9 & 0 & 1 & 1 & 2 \\ \hline\end{array} \tag{10.8}$$

• Case $(\mathbf{3},\mathbf{7},\mathbf{7})$: Condition (7.3) gives $q_1 = 3(p_1 - 1) \mid p_1 + 1$, i.e., $p_1 = 2$, but in this case $p_1 \geq 4$. Hence, there are no admissible triples of pairs.

• Case $(\mathbf{4},\mathbf{6},\mathbf{6})$: Here, and for the next three cases, $p_1 = \widehat{p}_1 = 6\xi_1 + 5$, $q_1 = 9\xi_1 + 6$ and $s_1 = \xi_1 + 1$ for an integer $\xi_1 \geq 0$. By (7.5) we have $\xi_1 + q_2 + q_3 \leq 11$. Condition (7.2) reads as

$$(9\xi_1 + 6)q_2 \mid (6\xi_1 + 5)q_2 + 9\xi_1 + 6 + q_3 = (9\xi_1 + 6)q_2 - (3\xi_1 + 1)q_2 + 9\xi_1 + 6 + q_3,$$

i.e.,

$$(9\xi_1 + 6)q_2 \mid (3\xi_1 + 1)q_2 - 9\xi_1 - 6 - q_3 \text{ with } (3\xi_1 + 1)q_2 - 9\xi_1 - 6 - q_3 \leq 0.$$

Since $1 \leq \xi_1 \leq 7$, we obtain

$$(3\xi_1 + 1)q_2 \leq 9\xi_1 + 6 + (11 - q_2 - \xi_1) \Longrightarrow (3\xi_1 + 2)q_2 \leq 17 + 8\xi_1 \leq 73,$$

i.e., $4 \leq (3\xi_1 + 2)q_2 \leq 73$. Since $2 \leq q_3 \leq 11 - (\xi_1 + q_2)$, the divisibility conditions (7.2) and (7.3) are satisfied only for $\xi_1 \in \{0, 1, 2\}$. In particular, for $\xi_1 = 0$ we obtain $(q_2, q_3) \in \{(8,2), (2,8)\}$ and the admissible triples of pairs

$$\begin{array}{|c|c|c|c|c|c|}\hline p_1 & q_1 & p_2 & q_2 & p_3 & q_3 \\ \hline 5 & 6 & 1 & 8 & 1 & 2 \\ \hline\end{array} \tag{10.9}$$

and

$$\begin{array}{|c|c|c|c|c|c|}\hline p_1 & q_1 & p_2 & q_2 & p_3 & q_3 \\ \hline 5 & 6 & 1 & 2 & 1 & 8 \\ \hline\end{array} \tag{10.10}$$

For $\xi_1 = 1$ we have necessarily $q_2 = q_3 = 5$ and the admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 11 | 15 | 1 | 5 | 1 | 5 |

Finally, for $\xi_1 = 2$ we have necessarily $q_2 = q_3 = 4$ and the admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 17 | 24 | 1 | 4 | 1 | 4 |

• Case $(\mathbf{4},\mathbf{6},\mathbf{7})$: By (7.5) we have $\xi_1 + q_2 \leq 10$. Condition (7.3) gives

$$9\xi_1 + 6 \mid 6\xi_1 + 5 + q_2 \Longrightarrow 9\xi_1 + 6 \mid 3\xi_1 + 1 - q_2, \text{ with } 3\xi_1 + 1 - q_2 \leq 0.$$

Thus, $3\xi_1 \leq q_2 - 1 \leq 9 - \xi_1 \Longrightarrow \xi_1 \leq \frac{9}{4} \Longrightarrow \xi_1 \in \{0,1,2\}$. Since $2 \leq q_2 \leq 10$, condition (7.3) implies

$$(\xi_1, q_2) \in \{(0,7), (1,4), (2,7)\}.$$

$(2,7)$ is not permitted because it violates (7.2); therefore, the admissible triples of pairs are

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 1 | 7 | 0 | 1 |

(10.11)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 11 | 15 | 1 | 4 | 0 | 1 |

(10.12)

• Case $(\mathbf{4},\mathbf{7},\mathbf{6})$: As in the case $(\mathbf{4},\mathbf{6},\mathbf{7})$ one proves that there are two admissible triples of pairs, namely

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 0 | 1 | 1 | 7 |

(10.13)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 11 | 15 | 0 | 1 | 1 | 4 |

(10.14)

• Case $(\mathbf{4},\mathbf{7},\mathbf{7})$: Conditions (7.2) and (7.3) give $q_1 = \frac{3}{2}(p_1 - 1) \mid p_1 + 1$, i.e., $p_1 = 5$. Thus, we find just one admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 5 | 6 | 0 | 1 | 0 | 1 |

• Case $(\mathbf{5},\mathbf{6},\mathbf{6})$: Here, and for the next three cases, $p_1 = 6\xi_1 + 1$, $\widehat{p}_1 = 3\xi_1 + 1$, $q_1 = 9\xi_1$ and $s_1 = \xi_1 + 1$ for an integer $\xi_1 \geq 1$. By (7.5) we have $\xi_1 + q_2 + q_3 \leq 11$. Condition (7.2) reads as

$$9\xi_1 q_2 \mid (3\xi_1 + 1)q_2 + 9\xi_1 + q_3 = 9\xi_1 q_2 - 6\xi_1 q_2 + q_2 + 9\xi_1 + q_3,$$

i.e.,

$$9\xi_1 q_2 \mid 6\xi_1 q_2 - q_2 - 9\xi_1 - q_3 \quad \text{with} \quad 6\xi_1 q_2 - q_2 - 9\xi_1 - q_3 \leq 0.$$

Since $1 \leq \xi_1 \leq 7$,

$$6\xi_1 q_2 \leq q_2 + q_3 + 9\xi_1 \leq 11 + 8\xi_1 \leq 67 \Longrightarrow 2 \leq \xi_1 q_2 \leq 11.$$

Since $2 \leq q_3 \leq 11 - (\xi_1 + q_2)$, the divisibility conditions (7.2) and (7.3) are satisfied only for $\xi_1 = 1$, $q_2 = 3$, $q_3 = 6$, or $\xi_1 = 2$, $q_2 = 2$, $q_3 = 4$, leading to two admissible triple of pairs, namely

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 7 | 9 | 1 | 3 | 1 | 6 |

(10.15)

and

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 13 | 18 | 1 | 2 | 1 | 4 |

(10.16)

• Case $(\mathbf{5},\mathbf{6},\mathbf{7})$: By (7.5) we have $\xi_1 + q_2 \leq 10$. Condition (7.3) gives

$$9\xi_1 \mid 6\xi_1 + 1 + q_2 \Longrightarrow 9\xi_1 \mid 3\xi_1 - q_2 - 1, \text{ with } 3\xi_1 - q_2 - 1 \leq 0.$$

Thus,

$$3\xi_1 \leq q_2 + 1 \leq 11 - \xi_1 \Longrightarrow \xi_1 \leq \frac{11}{4} \Longrightarrow \xi_1 \in \{1,2\}.$$

Since $2 \leq q_2 \leq 9$, condition (7.3) implies $\xi_1 = 1$ and $q_2 = 2$. The result is the following admissible triple of pairs:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 7 | 9 | 1 | 2 | 0 | 1 |

(10.17)

• Case $(\mathbf{5}, \mathbf{7}, \mathbf{6})$: By (7.5), $\xi_1 + q_3 \leq 10$. Now (7.2) reads as

$$9\xi_1 \mid 3\xi_1 + 1 + q_3 \Longrightarrow 9\xi_1 \mid 6\xi_1 - q_3 - 1, \text{ with } \; 6\xi_1 - q_3 - 1 \leq 0.$$

Thus,

$$6\xi_1 \leq q_3 + 1 \leq 11 - \xi_1 \Longrightarrow \xi_1 \leq \frac{11}{7} \Longrightarrow \xi_1 = 1.$$

Since $2 \leq q_3 \leq 9$, condition (7.2) implies $q_3 = 5$. The corresponding admissible triple of pairs is the following:

| $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ |
|---|---|---|---|---|---|
| 7 | 9 | 0 | 1 | 1 | 5 |

(10.18)

• Case $(\mathbf{5}, \mathbf{7}, \mathbf{7})$: Condition (7.3) gives $q_1 = \frac{3}{2}(p_1 - 1) \mid p_1 + 1$, i.e., $p_1 \leq 5$, but in this case $p_1 \geq 7$. Hence, there are no admissible triples of pairs. $\square$

**Remark 10.2.** The majority of the admissible triples of pairs induce toric log Del Pezzo surfaces admitting at least one Gorenstein singularity. This is due to the fact that the $q_i$'s corresponding to Gorenstein singularities can be viewed as parameters moving freely between 2 and an upper bound dictated by conditions (7.5) and (7.6), without any further restrictions.

## 11. Proof of Theorem 1.3: Step4

**Lemma 11.1.** *The toric log Del Pezzo surfaces induced by the following admissible triples of pairs* (**a**) *and* (**b**):

| (**a**) | (9.1) | (9.4) | (9.5) | (10.1) | (10.3) | (10.5) |
|---|---|---|---|---|---|---|
| (**b**) | (9.6) | (9.10) | (9.8) | (10.2) | (10.4) | (10.15) |

| (**a**) | (10.6) | (10.8) | (10.7) | (10.9) | (10.11) | (10.12) |
|---|---|---|---|---|---|---|
| (**b**) | (10.16) | (10.17) | (10.18) | (10.10) | (10.13) | (10.14) |

*are isomorphic to each other. The same is true for the four surfaces induced by the following admissible triples of pairs*:

| (**a**) | (**b**) | (**c**) | (**d**) |
|---|---|---|---|
| (9.2) | (9.3) | (9.7) | (9.9) |

(The admissible triples of pairs are given by their reference numbers.)

*Proof.* If $X_{\Delta_{(\mathbf{a})}}$ (resp., $X_{\Delta_{(\mathbf{b})}}$) is the toric Del Pezzo surface induced by the admissible triple of pairs (**a**) (resp., (**b**)) in the first list, then $\mathfrak{G}_{\Delta_{(\mathbf{a})}} \overset{\text{gr.}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_{(\mathbf{b})}}$. Correspondingly, if $X_{\Delta_{(\mathbf{a})}}, X_{\Delta_{(\mathbf{b})}}, X_{\Delta_{(\mathbf{c})}}, X_{\Delta_{(\mathbf{d})}}$ are the four surfaces induced by the admissible triples of pairs in the second list, then we obtain

$$\mathfrak{G}_{\Delta_{(\mathbf{a})}} \overset{\text{gr.}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_{(\mathbf{b})}} \overset{\text{gr.}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_{(\mathbf{c})}} \overset{\text{gr.}}{\cong} \mathfrak{G}_{\Delta_{(\mathbf{d})}}.$$

It is therefore enough to apply Theorem 4.4. $\square$

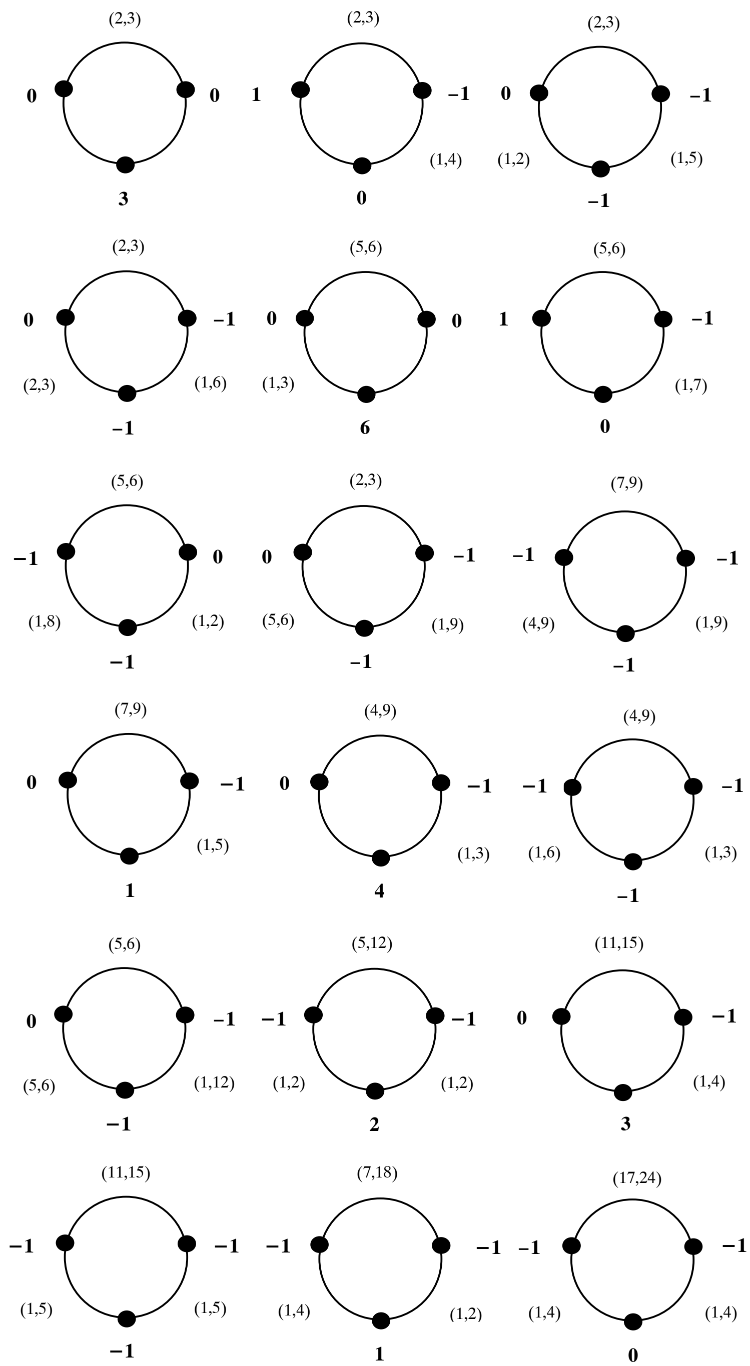
**Note 11.2.** By Lemmas 8.2, 9.1, 9.2 and 10.1 we proved that among all possible triples of pairs there exist exactly 33 which are admissible. Lemma 11.1 informs us that, in fact, for the classification of toric Del Pezzo surfaces $X_\Delta$ having Picard number $\rho(X_\Delta) = 1$ and index $\ell = 3$ *up to isomorphism*, we need only 18 out of them. (The $X_\Delta$'s induced by such a choice of 18 admissible triples of pairs are *obviously* pairwise non-isomorphic.)

*End of the proof of Theorem* 1.3: We consider 18 representatives of admissible triple of pairs inducing pairwise non-isomorphic toric Del Pezzo surfaces $X_\Delta$ with $\rho(X_\Delta) = 1$ and index $\ell = 3$, and we enumerate them, e.g., as in the Table 6. The coordinates of the third minimal generator $\mathbf{n}_3$ is computed by (6.2). The integers $r_i = -\overline{C}_i^2, i \in \{1, 2, 3\}$, are computed directly via (4.8).

| **No.** | **Case** | $p_1$ | $q_1$ | $p_2$ | $q_2$ | $p_3$ | $q_3$ | $\mathbf{n}_3$ | $r_1$ | $r_2$ | $r_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (i) | $(\mathbf{1},\mathbf{7},\mathbf{7})$ | 2 | 3 | 0 | 1 | 0 | 1 | $(-1,-1)$ | 0 | 0 | $-3$ |
| (ii) | $(\mathbf{1},\mathbf{7},\mathbf{6})$ | 2 | 3 | 0 | 1 | 1 | 4 | $(-3,-4)$ | 1 | $-1$ | 0 |
| (iii) | $(\mathbf{1},\mathbf{6},\mathbf{6})$ | 2 | 3 | 1 | 2 | 1 | 5 | $(-4,-5)$ | 1 | 0 | 1 |
| (iv) | $(\mathbf{1},\mathbf{1},\mathbf{6})$ | 2 | 3 | 2 | 3 | 1 | 6 | $(-5,-6)$ | 1 | 0 | 1 |
| (v) | $(\mathbf{4},\mathbf{7},\mathbf{7})$ | 5 | 6 | 0 | 1 | 0 | 1 | $(-1,-1)$ | 0 | 0 | $-6$ |
| (vi) | $(\mathbf{4},\mathbf{7},\mathbf{6})$ | 5 | 6 | 0 | 1 | 1 | 7 | $(-6,-7)$ | 1 | $-1$ | 0 |
| (vii) | $(\mathbf{4},\mathbf{6},\mathbf{6})$ | 5 | 6 | 1 | 8 | 1 | 2 | $(-3,-2)$ | 0 | 1 | 1 |
| (viii) | $(\mathbf{1},\mathbf{4},\mathbf{6})$ | 2 | 3 | 5 | 6 | 1 | 9 | $(-8,-9)$ | 1 | 0 | 1 |
| (ix) | $(\mathbf{5},\mathbf{3},\mathbf{6})$ | 7 | 9 | 4 | 9 | 1 | 9 | $(-8,-9)$ | 1 | 1 | 1 |
| (x) | $(\mathbf{5},\mathbf{7},\mathbf{6})$ | 7 | 9 | 0 | 1 | 1 | 5 | $(-4,-5)$ | 1 | 0 | $-1$ |
| (xi) | $(\mathbf{3},\mathbf{7},\mathbf{6})$ | 4 | 9 | 0 | 1 | 1 | 2 | $(-1,-2)$ | 1 | 0 | $-4$ |
| (xii) | $(\mathbf{3},\mathbf{6},\mathbf{6})$ | 4 | 9 | 1 | 6 | 1 | 3 | $(-2,-3)$ | 1 | 1 | 1 |
| (xiii) | $(\mathbf{4},\mathbf{4},\mathbf{6})$ | 5 | 6 | 5 | 6 | 1 | 12 | $(-11,-12)$ | 1 | 0 | 1 |
| (xiv) | $(\mathbf{2},\mathbf{6},\mathbf{6})$ | 5 | 12 | 1 | 2 | 1 | 2 | $(-1,-2)$ | 1 | 1 | $-2$ |
| (xv) | $(\mathbf{4},\mathbf{7},\mathbf{6})$ | 11 | 15 | 0 | 1 | 1 | 4 | $(-3,-4)$ | 1 | 0 | $-3$ |
| (xvi) | $(\mathbf{4},\mathbf{6},\mathbf{6})$ | 11 | 15 | 1 | 5 | 1 | 5 | $(-4,-5)$ | 1 | 1 | 1 |
| (xvii) | $(\mathbf{3},\mathbf{6},\mathbf{6})$ | 7 | 18 | 1 | 4 | 1 | 2 | $(-1,-2)$ | 1 | 1 | $-1$ |
| (xviii) | $(\mathbf{4},\mathbf{6},\mathbf{6})$ | 17 | 24 | 1 | 4 | 1 | 4 | $(-3,-4)$ | 1 | 1 | 0 |

TABLE 6.

The WVE$^2$C-graphs $\mathfrak{G}_\Delta$ (associated to the 18 $\Delta$'s) are depicted in Figure 3 in this order. (The reference to the double weight $(0, 1)$ at an edge of $\mathfrak{G}_\Delta$ is always omitted.) Finally, we may identify the corresponding $X_\Delta$'s with weighted projective planes or quotients thereof by a finite abelian group $H_\Delta$ via Lemma 6.1. (In the statement of the Theorem we have w.l.o.g. rearranged the weights in ascending order. Computing the Smith normal form, $H_\Delta$ turns out to be cyclic for the surfaces (ix) and (xviii)). $\square$


(2,3)
0
0
3
(2,3)
1
−1
(1,4)
0
(2,3)
0
−1
(1,2)
(1,5)
−1
(2,3)
0
−1
(2,3)
(1,6)
−1
(5,6)
0
0
(1,3)
6
(5,6)
1
−1
(1,7)
0
(5,6)
−1
0
(1,8)
(1,2)
−1
(2,3)
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−1
(5,6)
(1,9)
−1
(7,9)
−1
−1
(4,9)
(1,9)
−1
(7,9)
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(1,5)
1
(4,9)
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(1,3)
4
(4,9)
−1
−1
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(1,3)
−1
(5,6)
0
−1
(5,6)
(1,12)
−1
(5,12)
−1
−1
(1,2)
(1,2)
2
(11,15)
0
−1
(1,4)
3
(11,15)
−1
−1
(1,5)
(1,5)
−1
(7,18)
−1
−1
(1,4)
(1,2)
1
(17,24)
−1
−1
(1,4)
(1,4)
0

Figure 3.

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